PARALLEL FOCAL STRUCTURE AND SINGULAR RIEMANNIAN FOLIATIONS

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ABSTRACT. We give a necessary and sufficient condition for a submanifold with parallel focal structure to give rise to a global foliation of the ambient space by parallel and focal manifolds. We show that this is a singular Riemannian foliation with complete orthogonal transversals. For this object we construct an action on the transversals that generalizes the Weyl group action for polar actions.

1. INTRODUCTION

In his thesis [Ew], Ewert introduced the notion of a submanifold with parallel focal structure as a generalization of isoparametric submanifolds in Euclidean space (see for instance [PaTe]) and equifocal submanifolds in simply connected symmetric spaces ([TeTh]). For a survey on these and related objects, see [Th2]. In order to generalize results known for isoparametric and equifocal submanifolds we pursue a completely different approach than in the respective theories. We first look for a condition under which the given submanifold with parallel focal structure gives rise to a singular foliation of the ambient space. Then we will obtain similar properties as for isoparametric and equifocal submanifolds as a consequence of this foliated structure.

Let N be a complete Riemannian manifold and M be a submanifold. We begin by asking under which conditions we have a good partition of N by parallel and focal manifolds of M. To properly define parallel and focal manifolds, M has to suffice the following minimal conditions.

(1) νM is flat, i.e., any $v \in \nu M$ can be locally extended to a parallel normal field v'.
(2) the rank of the locally defined map $\exp \circ v'$ is constant for any $v \in \nu M$.

For $v \in \nu M$ we define

$$M_v = \left\{ \exp\left((\overset{1}{\underset{0}{\|}} c) v \right) \;\middle|\; c \text{ is a curve in } M \right\},$$

where $\| c$ denotes normal parallel translation along c. We call M_v *parallel manifold* of M if the rank of $\exp \circ v'$ is maximal, otherwise *focal manifold*.

Definition 1.1. ([HLO]) We say that M gives rise to a *global foliation* $\mathcal{F} = \{M_v \mid v \in \nu_p M\}$ of N, if $\bigcup \mathcal{F} = N$, and $M_v \cap M_w \neq \emptyset$ implies $M_v = M_w$.

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Example 1.2. First we take $N = S^2$ and M a parallel of the equator. Clearly M induces a global foliation. Next we consider the flat torus $N = T^2$ and a small distance circle M centered at a point p in N. M does not induce a global foliation of N.

With the notion of the cut locus of a proper immersion we give a necessary condition for M under the above minimal conditions to induce a global foliation in Proposition 2.4. This condition is not sufficient. Now assume that M fulfills the minimal conditions and in addition that through any $x \in M$ there is a complete totally geodesic submanifold Σ_x, a *section*, with $T_x\Sigma_x = \nu_x M$. This class of submanifolds generalizes isoparametric submanifolds in Euclidean space and equifocal submanifolds in simply connected symmetric spaces for arbitrary ambient spaces. For this class we prove that the condition, that any two sections through a regular value of the normal exponential map of M coincide, is necessary and sufficient to give rise to a global foliation. We then call M a *submanifold with parallel focal structure* (the precise definition is given in 3.4). We only formulate the sufficiency condition:

Theorem. *A closed and embedded submanifold M with parallel focal structure and finite normal holonomy of a complete Riemannian manifold gives rise to a global foliation $\mathcal{F}$. This global foliation is a singular Riemannian foliation admitting sections.*

The definition of a singular Riemannian foliation is given below. The converse statement, that a regular leaf of a singular Riemannian foliation admitting sections has parallel focal structure was proven by Alexandrino in [A].

Definition 1.3 ([Mo])**.** Let $\mathcal{F}$ be a partition of injectively immersed submanifolds (the *leaves*) of a Riemannian manifold N. For any $p \in N$ let M_p be the leaf through p and let $T\mathcal{F} = \bigcup_{p\in N} T_pM_p$. We define $\Xi(\mathcal{F})$ as the module of (differentiable) vector fields on N with values in $T\mathcal{F}$. We call $\mathcal{F}$ a *singular Riemannian foliation*, if

1. (Transnormality) a geodesic starting orthogonally to a leaf intersects the leaves it meets orthogonally;
2. (Differentiability) $\Xi(\mathcal{F})$ acts transitively on $T\mathcal{F}$, i.e., for any $v \in T_p\mathcal{F}, p \in N$ there is $X \in \Xi(\mathcal{F})$ with $X_p = v$.

A leaf of maximal dimension is called *regular*, and so each point of it, otherwise *singular*. If, in addition, for any regular p there is an isometrically immersed complete totally geodesic submanifold Σ_p (the *section*) with $T_p\Sigma = \nu_pM_p$, that meets any leaf and always orthogonally, $\mathcal{F}$ is a singular Riemannian foliation *admitting sections.*

A partition into injectively immersed submanifolds with every leaf of the same dimension is a foliation if and only if (2) is fulfilled. A foliation with (1) is a Riemannian foliation. The set of orbits of an isometric Lie group action on a Riemannian manifold N is a singular Riemannian foliation. The set of orbits of a polar action is a singular Riemannian foliation admitting sections.

In section 2 we introduce the cut locus of a submanifold.

In section 3 we prove the above theorem. In 3.1 we describe general properties of submanifolds with parallel focal structure. In 3.2 we associate a regular Riemannian foliation $(\hat{N}, \hat{\mathcal{F}})$ to M, the *blow-up*, which was constructed in [Bou] for

singular Riemannian foliations admitting sections (with relatively compact leaves). With this blow-up we show that each parallel manifold of M also has parallel focal structure and we derive the theorem in 3.3.

In section 4 we study singular Riemannian foliations. In 4.1 we give an alternative proof of the converse of the above theorem. In 4.2 we introduce an action on the sections, the *transversal holonomy group*, which generalizes the Weyl group action of a polar action and we give applications. Furthermore we prove that each regular leaf of a singular Riemannian foliation admitting sections in a simply connected symmetric space has trivial normal holonomy.

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2. Cut Locus of a Proper Immersion

In this section we introduce the notion of a cut locus of a submanifold. This is a generalization of the cut locus of a point which is defined for instance in [Kl]. With the aid of this notion we can formulate a necessary condition for a properly immersed submanifold with the two minimal conditions stated in the first section to give rise to a global foliation.

Let N be a complete and connected Riemannian manifold and M a manifold. By $\varphi : M \to N$ we will always denote an isometric immersion. Let $\iota : \nu M \to TN$ be the canonical immersion. We write $\eta := \exp^{\perp} = \exp \circ \iota : \nu M \to N$ and $\eta^r : B_r(\nu M) \to N$ for the restriction of η to the normal ball bundle of M of radius r. Let $\varphi : M \to N$ be a proper immersion. Let γ_v denote the geodesic with initial vector $v \in TN$.

Definition 2.1. We define $\sigma : \nu^1 M \to [0, \infty]$ by

$$\sigma(v) = \sup\{t \in \mathbb{R} \mid d(\gamma_{\iota(v)}(t), \varphi(M)) = t\}.$$

We call $\sigma(v)$ the *cut distance of M in direction v*. The *normal cut locus* $\mathcal{C}^{\nu M}$ of φ is defined by $\mathcal{C}^{\nu M} := \{\sigma(v)v \mid \sigma(v) < \infty, v \in \nu^1 M\}$ and the *cut locus* $\mathcal{C}_\varphi$ or $\mathcal{C}_{(M,N)}$ by $\exp^{\perp} \mathcal{C}^{\nu M}$.

Definition 2.2. A vector $v \in \nu M$ is called a *normal* or *normal vector* and the geodesic $\gamma_{\iota(v)}$ a *normal geodesic*. If $\|v\| \leq \sigma(v/\|v\|)$ for $v \in \nu M$, we call v *minimal* and $\gamma_v|[0, 1]$, and any reparametrization of constant speed, a *minimal geodesic (segment)*. This terminology is justified by the fact that, in the set $\eta^{-1}(p)$, the minimal vectors have the least length. We call a normal vector v a *focal normal* and $\eta(v)$ a *focal point*, if v is singular with respect to η. We call a minimal vector v a *cut vector* and $\eta(v)$ a *cut point*, if there is a minimal $w \in \nu M$ with $\iota(w) \neq \iota(v)$ having the same endpoint as v. In this case $\|v\| = \sigma(v/\|v\|) = \sigma(w/\|w\|) = \|w\|$.

It is easy to see that the limit of a converging sequence of minimal normal vectors is minimal and it is kown that tv for $t > 1$ is not minimal if v is a focal vector or a cut vector. Also note that for every $p \in N$ there is a shortest curve from $\varphi(M)$ to p since $\varphi(M)$ is closed and N complete; this is a normal geodesic. This implies that η is surjective.

In contrast to the cut locus of a point the cut distance function is in general not continuous. It is upper semi-continuous and it is discontinuous, but not necessarily lower semi-continuous. In analogy to the cut locus of a point we have the following result:

Proposition 2.3. *The cut locus only consists of focal and cut points.*

Proof. We consider a $v \in \nu^1 M$ with $\sigma(v) < \infty$ such that $\sigma(v)v$ is not a focal normal. We have to show that $\eta(\sigma(v)v)$ is a cut point. We construct sequences $(\tilde{v}_n)$ in $\nu^1 M$ and $t_n > 0$ such that $t_n\tilde{v}_n$ is minimal with $\eta(t_n\tilde{v}_n) = \eta((\sigma(v)+1/n)v)$. As $\eta|\bar{B}_r(\nu M)$ is proper for all $r \geq 0$ we can assume that $t_n\tilde{v}_n$ converges to, say $t_0\tilde{v}$, where $\tilde{v} \in \nu^1 M$. Then $\sigma(v)v$ and $t_0\tilde{v}$ are minimal and have the same endpoint. This implies $t_0 = \sigma(v)$. Since η is injective on a neighborhood of $\sigma(v)v$ we have $\iota(\tilde{v}) \neq \iota(v)$ and $\eta(\sigma(v)\tilde{v}) = \eta(\sigma(v)v)$. □

Later we will frequently us the following notion. If $r = \inf\{\sigma(v) \mid v \in \nu^1 M\} > 0$ we call r *injectivity radius* of φ and $T_s = tube(M, s) = \exp B_s(\nu M)$ an *injectivity tube* of M with radius s for any s with $0 < s \leq r$. By definition for each $p \in T$ there is exactly one minimal normal v with endpoint p up to foot point. If φ is injective, $\eta : B_s(\nu M) \to T_s$ is a diffeomorphism. With the aid of the cut locus we can now already give a necessary condition for a properly immersed submanifold M with the two minimal conditions to give rise to a global foliation of the ambient space N.

Proposition 2.4. *Suppose a (topologically) closed submanifold M satisfying the minimal conditions given in the introduction induces a global foliation $\mathcal{F}$. Then the cut distance function is constant along parallel normal fields.*

Proof. Assume that M induces a global foliation and that the cut distance function σ is not constant for some parallel normal field. Then there is a $v_0 \in \nu M$ with $\|v_0\| > \sigma(v_0/\|v_0\|)$ and a parallel translation $v_1 \in \nu M$ with $\|v_1\| < \sigma(v_1/\|v_1\|)$, i.e. v is minimal. We find a minimal vector $w_0 \in \nu M$ with $\eta(w_0) = \eta(v_0)$. Since $M_v = M_w$ there is a normal parallel translation w_1 of w_0 with $\eta(w_1) = \eta(w_2)$. So $\|w_0\| < \|v_0\| = \|v_1\| \leq \|w_1\|$ in contradiction with $\|w_0\| = \|w_1\|$. □

Indeed the condition of the proposition is not fulfilled in the second example in 1.2. This condition is general not sufficient. The search for a necessary and sufficient condition can be answered for the class of submanifolds with sections. We will do this in the next section.

3. Submanifolds with Parallel Focal Structure

3.1. **General Properties.** Let M and N be complete and connected Riemannian manifolds and $\varphi : M \to N$ an isometric immersion. The aim of this section is to find minimal conditions for M to foliate the ambient space N with parallel submanifolds.

For any $v \in T_x N$ we have a decomposition of $T_v TN$

$$T_v TN = H_v^M \oplus V_v^N \cong T_x N \times T_x N$$

into the horizontal space H_v^N and the vertical space V_v^N with respect to the Levi-Civita connection ∇. The pullback of the metric on N with the canonical isomorphism $H_v^M \oplus V_v^N \cong T_x N \times T_x N$ is the Sasaki-metric by definition. Let M be a submanifold of N. Similar as above, for any $v \in \nu M$ we obtain a decomposition

$$T_v \nu M = H_v^M \oplus V_v^M \cong T_x M \oplus \nu_x M$$

into the horizontal space H_v^M and the vertical space V_v^M with respect to the normal Levi-Civita connection $\nabla^\perp$. Obviously $V^M \subset V^N$ but in general we do not have $H^M \subset H^N$. Indeed, an element $\xi = (\xi_h, \xi_v) \in T_v \nu M$ is equal to $(\xi_h, \xi_v - A_v \xi_h)$ as an element of $T_v TN$, where A is the shape operator of M.

We have the isomorphism between T_vTN and the vector space of Jacobi fields of N along γ_v mapping an element $\xi \in T_vTN$ to the Jacobi field $J = J_\xi$ given by $(J(t), J'(t)) = \phi^t_*(\xi_h, \xi_v)$, where ϕ^t is the time t map of the geodesic flow $\phi : \mathbb{R} \times TN \to TN$. The inverse map is given by $J \mapsto (J(0), J'(0))$. The restriction of the first map to $T_v\nu M$ is an isomorphism onto the vector space $\mathcal{J}_M(v)$ of M-Jacobi fields along γ_v

$$T_v\nu M \to \mathcal{J}_M(v); \quad \xi \mapsto J_\xi \quad \text{with} \quad (J_\xi(t), J'_\xi(t)) = \phi^t_*(\xi_h, \xi_v - A_v\xi_h).$$

The inverse map is given by $J \mapsto (J(0), J'(0)^\perp)$. The decomposition $T_v\nu M = H^M_v \oplus V^M_v$ carries over to the decomposition of $\mathcal{J}_M(v)$ into a horizontal and a vertical subspace. We can describe a vertical/horizontal M-Jacobi field J with initial condition $\xi \in T_v\nu M$ by a variational vector field. Define $V(s,t) = \eta(tX(s))$, where X is a vector field along the constant curve $c \equiv x$ with $\frac{dX}{dt}|_{t=0} = \xi_v$ if J is vertical, and a parallel normal field along c in M with $\dot c(0) = \xi_h$ if J is horizontal. Then $J(t) = \partial_s V(0,t)$.

Definition 3.1. Let $\varphi : M \to N$ be an immersion, $\iota : \nu M \to TN$ be the canonical inclusion. For $x \in M$ we call an isometric immersion $i_x : \Sigma_x \to N$ (or shorter Σ_x) with $(i_x)_*(T_x\Sigma_x) = \iota(\nu_x M)$ a *section*, if it is totally geodesic in N and if Σ_x is complete. Note that if we compose i_x with a covering onto Σ_x, we obtain another section. Since we want i_x to be unambiguous, we also demand $y = z$ whenever $(i_x)_*(T_y\Sigma_x) = (i_x)_*(T_z\Sigma_x)$. The immersion $\varphi : M \to N$ is said to *admit sections* if Σ_x is a section for every $x \in M$ and if there is exactly one section of φ through every regular point of the normal exponential map, i.e. if $p \in i_x(\Sigma_x) \cap i_y(\Sigma_y)$ is regular then $i_x = i_y \circ \alpha$ for some isometry $\alpha : \Sigma_x \to \Sigma_y$.

In order to avoid a cumbersome notation, we use Σ_x and the term section in two different ways. When it comes to point sets, for instance, if we write $p \in \Sigma_x$, we actually mean by Σ_x the image of the immersion i_x. If we talk about tangent vectors or curves of Σ_x, i.e., if the context is a topological or differentiable one, we are of course referring to the underlying manifold structure of the section. This distinction is particularly important here, since we allow Σ_x to have self-intersections.

Lemma 3.2. *Let γ be a geodesic in a section $\Sigma = \Sigma_x$ with $\gamma(0) = p = \varphi(x)$. Then any Jacobi field in N along γ can be decomposed into $J = J_1 + J_2$, where J_1 is a Jacobi field of Σ and J_2 is Jacobi field with $J_2(t) \in T_{\gamma(t)}\Sigma^\perp$ for every t. For an M-Jacobi field J this decomposition is exactly the one into vertical and horizontal M-Jacobi fields along γ. In particular we have $J(t) \perp T_{\gamma(t)}\Sigma$ for a horizontal M-Jacobi field J.*

Proof. We write J_1 for the $T\Sigma$-part of J and J_2 for the orthogonal part. Since Σ is totally geodesic, the curvature operator $R_{\dot\gamma(t)}$ leaves $T_{\gamma(t)}\Sigma$ invariant and therefore, as a self-adjoint operator, also the orthogonal complement $T_{\gamma(t)}\Sigma^\perp$, so $R_{\dot\gamma(t)}J_1(t) \in T_{\gamma(t)}\Sigma$ and $R_{\dot\gamma(t)}J_2(t) \in T_{\gamma(t)}\Sigma^\perp$. On the other hand we have $J_1''(t) \in T_{\gamma(t)}\Sigma$, since Σ is totally geodesic, and $J_2''(t) \in T_{\gamma(t)}\Sigma^\perp$ for all t, because of $0 = \frac{d^2}{dt^2} g(J_2(t), X(t)) = g(J_2''(t), X(t))$ for any parallel field X of Σ along γ. The Jacobi identity for J gives

$$0 = R_{\dot\gamma(t)}J(t) + J''(t) = (R_{\dot\gamma(t)}J_1(t) + J_1''(t)) + (R_{\dot\gamma(t)}J_2(t) + J_2''(t)).$$

Since the term in the first bracket lies in $T_{\gamma(t)}\Sigma$ and the term in the second in $T_{\gamma(t)}\Sigma^{\perp}$, the vector fields J_1 and J_2 are also Jacobi fields. The second statement follows from the initial conditions $(J_i(0), J_i'(0)^{\perp})$ of J_i for $i = 1, 2$. $\square$

The kernel of $d\eta(v)$ consists of $(J(0), J'(0)^{\perp})$, where J is an M-Jacobi field along γ_v with $J(1) = 0$. The decomposition $J = J_1 + J_2$ as in the lemma then implies that $\ker d\eta(v)$ is a direct sum of a horizontal and a vertical subspace of $T_v\nu M$ and that the kernel of $d\eta(v)$ only has a non-trivial vertical component if and only if $\eta(v)$ is a conjugate point of x along γ_v in Σ_x. Summing up, the decomposition of an M-Jacobi field J into $J = J_1 + J_2$ means that

$$\begin{array}{rcl} d\exp^{\perp}(v) : H_v^M \oplus V_v^M & \rightarrow & T_{\eta(v)}\Sigma^{\perp} \oplus T_{\eta(v)}\Sigma \\ (J(0), J'(0)^{\perp}) & \mapsto & J_1(1) + J_2(1) \end{array} \tag{3.1}$$

splits as an orthogonal direct sum of linear maps $H_v^M \rightarrow T_{\eta(v)}\Sigma^{\perp}$ and $V_v^M \rightarrow T_{\eta(v)}\Sigma$. We call this *splitting of* η.

Definition 3.3. We call a focal normal v of *horizontal/vertical type* if $\ker d\eta(v)$ has a non-trivial horizontal/vertical component. If a normal vector v is not a focal normal of horizontal type we call v *f-regular*. A point $p \in N$ is called *f-regular* if there is an f-regular normal v such that $\eta(v) = p$. For a normal vector $v \in \nu_x M$ we call the dimension of the horizontal factor of $\ker d\eta(v)$ the *horizontal multiplicity* of v. Note that we have slightly changed the definitions given in [Ew].

Definition 3.4. An immersion $\varphi : M \rightarrow N$ has *parallel focal structure*, if

(1) νM is flat,
(2) $\dim(\ker d\eta(v) \cap H_v^M) = \dim \ker d(\eta \circ v)$ is constant for any local parallel normal field v, i.e. the horizontal focal data is invariant under normal parallel translation, and
(3) φ admits sections.

In contrast to [Ew] we do not demand the invariance of the vertical data. We will show in Proposition 4.22 that this second invariance is an implication.

Example 3.5. Regular orbits of polar actions have parallel focal structure. Isoparametric submanifolds in $\mathbb{R}^{n+k}$ and equifocal submanifolds in simply connected, compact symmetric spaces obviously fulfill conditions (1) and (2) of a submanifold. The existence of sections for both classes of submanifolds follows from the properties of the respective ambient space. Theorem 3.17 will show, that they admit sections if and only if the set of parallel manifolds builds a foliation on the regular set, which is known for both classes.

We assume that φ admits sections and that νM is flat. We define two distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ on the set N_r of f-regular points in N by $\mathcal{D}^{\perp}(p) = T_p\Sigma$, where Σ is a section through p; let $\mathcal{D}$ be the orthogonal distribution. The distribution $\mathcal{D}^{\perp}$ and therefore $\mathcal{D}$ are well-defined on the set of regular points, since M admits sections, but a priori not on the set of f-regular points. It is easy to see that both distributions are integrable on the regular set: Let p be a regular point and $v \in \nu M$ with $\eta(v) = p$. Recall that νM carries the horizontal foliation $\mathcal{P}$ given by normal parallelity, and the vertical foliation $\mathcal{P}^{\perp}$ given by the fibers of the projection $\nu M \rightarrow M$. Now let U_v be an open neighborhood of $v \in \nu M$ such that $\eta|U_v : U_v \rightarrow V$ from U_v onto its image V is a diffeomorphism. The map $\eta|U_v$ maps vertical leaves diffeomorphically onto open subsets of sections. The splitting of η says that $d\eta$ maps the

horizontal distribution on νM to $\mathcal{D}$, i.e. $d\eta(v)(T_xM) = \mathcal{D}(\eta(v))$. Since U_v is bifoliated and $\eta|U_v$ is a diffeomorphism, V is also bifoliated with respect to $\mathcal{D}$ and $\mathcal{D}^\perp$. We want to show that both distributions are also differentiable and well-defined on the set N_r of f-regular points in N. Integrability is clear.

Lemma 3.6. *There is exactly one section Σ through a given f-regular point p and $\eta^{-1}(p)$ only consists of f-regular vectors that are tangential to Σ. Moreover, N_r is open and dense in N and there is a unique differentiable extension of $\mathcal{D}^\perp$ on N_r. The distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ give rise to a bifoliation $(\mathcal{F}_r, \mathcal{F}_r^\perp)$ of N_r.*

Proof. Existence follows by surjectivity of η. We show uniqueness. Let $v_0 \in \nu_x M$ be an f-regular vector with $\eta(v_0) = p$. Then there is a simply connected neighborhood U of x in M such that $(\eta\circ v)|U : U \to P_{v_0} = \eta(U)$ is a diffeomorphism, where v is a parallel normal field on U with $v_x = v_0$. We define $T = tube(P_{v_0}, \varepsilon) = \{\exp(\xi) \mid \xi \in B_\varepsilon(\nu P_{v_0})\}$. By shrinking U we can assume that T is an injectivity tube around P_{v_0} for small $\varepsilon > 0$. Let $\rho : T \to P_{v_0}$ be the projection. We have $T_{\eta(v_z)}\Sigma_z \perp T_{\eta(v_z)}P_{v_0}$ for every $z \in U$ by the splitting of η. Therefore *a slice of the tube T through $\eta(v_z) \in P_{v_0}$ coincides with the component of $\Sigma_z \cap T$ containing $\eta(v_z)$.* We can therefore extend $\mathcal{D}^\perp$ differentiably to T as the kernel of the differential of the submersion ρ. Since $\mathcal{D}^\perp$ is defined on the open and dense set of regular points of N, *this extension is the unique differentiable extension of $\mathcal{D}^\perp$.*

Let $w_0 \in \nu_y M$ be another f-regular vector with $\eta(w_0) \in T$. The same process as for v_0 gives us a simply connected neighborhood U' of y, a parallel normal field w extending w_0, P_{w_0} and its tube T' with the same properties. By eventually shrinking U' and the radius of T' we can assume $T' \subset T$. By the uniqueness of a differentiable extension of $\mathcal{D}^\perp$ we conclude that the slices of T' are equal to the slices of T intersected with the open set T'. In particular, if $\eta(w_0) = p$ this implies that v_0 and w_0 are tangential to the same section $\Sigma_x = \Sigma_y$. Since w_0 is f-regular, $\eta \circ w$ has maximal rank on a neighborhood of y. We can assume this neighborhood to be U'. Then P_{w_0} intersects the slices transversally, i.e. $\rho \circ \eta \circ w : U' \to P_{v_0}$ is a diffeomorphism onto its image $(*)$.

We have seen above that the f-regular vectors in $\eta^{-1}(p)$ are tangential to the same section. Now we are going to show that any $w_0 \in \nu M$ with $\eta(w_0) \in T$ is f-regular. Then $T \cap \Sigma$ is an open neighborhood of p in Σ only containing f-regular points. This implies that the set of f-regular points is open in N and that $\eta^{-1}(p)$ only consists of f-regular vectors. We remark that this even shows that the f-regular points in a section Σ are open in Σ (see Corollary 3.7). Let $w_0 \in \nu_y M$ with $\eta(w_0) \in T$ and U' a neighborhood of w_0 in νM such that $\eta(U') \subset T$. We want to show that w_0 is f-regular. We can locally define a parallel normal field w extending w_0. Then there is a simply connected neighborhood U of y in M and an $\varepsilon > 0$ such that the image of U under $(1 + t)w$ lies in U' for all $t \in (0, \varepsilon)$ and such that w'_z is f-regular for every $z \in U$, where $w' = (1 + \varepsilon)w$. The geodesic $\gamma_{w(z)}$ intersects $P_{(1+\varepsilon)w_0}$, the image of $\eta \circ w'$, orthogonally in $\gamma_{w(z)}(1 + \varepsilon)$ for all $z \in U$ by the splitting of η or the Gauss Lemma for the normal exponential map. Then the image of $\gamma_w|[1, 1+\varepsilon]$ lies in a slice of the tube T. Therefore $\rho \circ \eta \circ w = \rho \circ \eta \circ w'$ on U. Since the right side is a diffeomorphism $(*)$ this implies that also $\eta \circ w$ has maximal rank, i.e. w_0 is f-regular.

For any f-regular point p we obtain a neighborhood T as above that is bifoliated with respect to $\mathcal{D}$ and $\mathcal{D}^\perp$. $\square$

The lemma says that the preimage of a focal point $\eta(v)$, where v is a focal normal of horizontal type, only consists of focal normals of horizontal type.

By the Theorem of Sard the set of regular points of η is open and dense in N. Obviously the intersection of the set of regular points with Σ is open in Σ. It is a priori not clear that the set of regular points in Σ is dense in Σ. That this is true says the following corollary of the last lemma.

Corollary 3.7. *The subset of f-regular points in a section Σ is open and dense in Σ.*

Proof. We have seen in the proof of Lemma 3.6 that the subset of f-regular points in Σ is open in Σ. For the proof of density, see [Tö]. □

Now let φ be submanifold with parallel focal structure. Then every horizontal leaf L_v in νM through an f-regular vector v is contained in $\eta^{-1}(N_r)$. The map $\eta : L_v \to M_v$ is a covering. Obviously M_v is open in a leaf. Using completeness of M one can easily see that M_v is a leaf. The first statement of the following proposition follows from Lemma 3.6.

Proposition 3.8. *Let φ be a submanifold with parallel focal structure. Then the leaves of $\mathcal{F}_r^{\perp}$ are the parallel manifolds and the leaves of $\mathcal{F}_r^{\perp}$ the components of the restrictions of the sections to N_r. $\mathcal{F}_r^{\perp}$ is a totally geodesic foliation and therefore $\mathcal{F}_r$ a Riemannian foliation. Moreover, every parallel manifold has a flat normal bundle.*

Proof. The characterization of the leaves of $\mathcal{F}_r$ is given above, the one for leaves of $\mathcal{F}_r^{\perp}$ is clear. By definition the sections are totally-geodesic, therefore $\mathcal{F}_r^{\perp}$ is a totally geodesic foliation. Let us consider a bifoliation $(\mathcal{F}_1, \mathcal{F}_2)$ of a Riemannian manifold, where the two foliations are orthogonal to each other. It is well-known that $\mathcal{F}_1$ is a Riemannian foliation if and only if $\mathcal{F}_2$ is a totally geodesic foliation. Let us consider this kind of bifoliation. Let L be a leaf of $\mathcal{F}_1$ and $v \in \nu L$ with footpoint p. We consider a plaque P through p of a neighborhood U of p that is simple (or foliated) with respect to $\mathcal{F}_1$. There is a vector field U tangential to $\mathcal{F}_2$ and foliated with respect to $\mathcal{F}_1$ extending v. The restriction of this vector field to P is a parallel normal field of P. Therefore any leaf of $\mathcal{F}_1$ and in particular our parallel manifolds have a flat normal bundle. For more details see [Tö]. □

Let $\bar{M}$ be the normal holonomy principal bundle over M equipped with the metric such that the projection $\bar{M} \to M$ becomes a Riemannian covering. Its normal bundle is globally flat and $\bar{M} \to M$ has the lowest degree among all coverings of M with this property. Each normal vector v of M canonically defines a global parallel normal field on $\bar{M}$, denoted by $\bar{v}$. We will denote the normal exponential map of $\bar{M}$ also by η. Proposition 3.8 implies that φ and $\eta \circ \bar{v}$ for f-regular v factorize through injective immersions, the first even through an injective isometric immersion. If φ is proper it factorizes finitely over an embedding; if in addition v is f-regular and has finite normal holonomy degree then $\eta \circ \bar{v} : \bar{M} \to N$ is also a proper immersion, since η restricted to $\bar{B}_r(\nu M) = \{w \in \nu M \mid \|w\| \leq r\}$ is proper for any $r \geq 0$. So from now on we can assume that φ is injective and the inclusion map of M into N. Let us repeat the definition of parallel and focal manifolds.

Definition 3.9. Let $\varphi : M \to N$ have parallel focal structure. We call $\eta \circ \bar{v} : \bar{M} \to N$ a *focal submanifold* of M if $v \in \nu M$ is a focal normal of horizontal type, a *parallel submanifold*, if v is f-regular. In any case we denote the image by M_v.

Let v be a focal normal of horizontal type. Since the map $\eta \circ \bar{v}$ has constant rank, the set of connected components of preimages of $\eta \circ \bar{v}$ defines a foliation (the *focal foliation*) by the rank theorem which gives us simple sets for this foliation. The leaf through x is called the *focal leaf* $F_{\bar{v}_x}$ through x associated to v (or to $\bar{v}_x$). If $\mathcal{G}$ denotes the focal foliation, $\bar{M}/\mathcal{G}$ endowed with the quotient topology is not necessarily Hausdorff or second countable but carries a natural differentiable structure by Theorem VIII of [Pa] for which the map $\eta \circ \bar{v} : \bar{M} \to N$ induces an immersion $M/\mathcal{G} \to N$. Thus M_v is the image of an immersion.

3.2. **The Blow-Up $(\hat{N}, \hat{\mathcal{F}}, \hat{\mathcal{F}}^{\perp})$.** Each parallel manifold has the same set of sections as M and thus we have a splitting of its normal exponential map. By a similar argument as in Lemma 3.6, we can show that each parallel submanifold M' has the same set of f-regular points in N, namely N_r, and therefore admits sections by Proposition 3.8. The restriction $\eta'|L$ of the normal exponential map η' of a parallel manifold M' to a horizontal leaf L' of the flat bundle $\nu M'$ through an f-regular vector is a covering map onto a leaf of $\mathcal{F}$. In order to show that M' has parallel focal structure it remains to show that $\eta'|L$ has constant rank, if L' is a horizontal leaf in $\nu M'$ through a focal normal of horizontal type. We will see this in Proposition 3.14.

Our main goal in this section is to show first that $\mathcal{F} = \{M_v \mid v \in \nu M\}$ is a global foliation and then a singular Riemannian foliation. In this subsection we will associate to $(N, \mathcal{F})$ a certain foliated manifold $(\hat{N}, \hat{\mathcal{F}})$. An analysis of this foliation will yield the results. Boualem defines this Riemannian foliation $\hat{\mathcal{F}}$ in [Bou] from a singular Riemannian foliation $\mathcal{F}$. Thus we cannot use his construction. Instead we build up $\hat{\mathcal{F}}$ with the normal exponential map.

For an f-regular point $x \in N$ let $\eta_x : \nu M_x \to N$ be the normal exponential map of the leaf M_x. We define

$$\hat{\eta}_x : \nu M_x \to G_k(TN); v \mapsto T_{\eta_x(v)}\Sigma_y,$$

where y is the footpoint of v. Note that Σ can have self-intersections in focal points of horizontal type. Therefore we only have a well-defined tangential space $T_p\Sigma$ in f-regular points p, so our above definition of $\hat{\eta}_x$ is not precise. A correct definition is as follows. Let $i : \Sigma_y \to N$ be the unique section through y. We identify any $z \in \Sigma_y$ with its image $i(z)$, if $i(z)$ is f-regular. We define $\hat{\eta}_x(v) := i_*(T_{\gamma_v(1)}\Sigma_y)$ where γ_v is the geodesic in Σ_y with initial vector v. Since this notation is too cumbersome, we prefer the first one, but we have to keep in mind that the expression $T_{\eta(v)}\Sigma_y$ depends on v and not only on $\eta(v)$. Let

$$\hat{N} = \{T_q\Sigma \mid \Sigma \text{ is a section}, q \in \Sigma\}.$$

and let $\hat{\pi} : \hat{N} \to N$ be the footpoint map of $G_k(TN)$ restricted to $\hat{N}$. Then we have $\hat{N} = \hat{\eta}_x(\nu M_x)$ for any f-regular point $x \in N$ since the set of sections of two different parallel manifolds coincide. Also note that $\eta_x = \hat{\pi} \circ \hat{\eta}_x$ Our next aim is to give a bifoliated manifold structure to $\hat{N}$. The idea is to model $\hat{N}$ on the normal bundles of the parallel submanifolds, the charts being the maps $\hat{\eta}_x$. The normal bundle νM has two natural, complemetary foliations $\mathcal{P}$ and $\mathcal{P}^{\perp}$, one given by the flat horizontal structure, the other by the fibers of the projection $\nu M \to M$.

Let $p \in N$ be arbitrary. We fix $r > 0$ and take $\varepsilon' > 0$ to be smaller than the injectivity radius of any point $q \in \bar{B}_r(p)$ in N. There is an f-regular point x and a vector $v \in \nu_x M_x$ with $\eta_x(v) = p$ that is not a focal normal of vertical type. One

can see that $d\hat\eta_x(w)|\mathcal{H}^M_w$ is injective for any $w \in \nu M_x$. Therefore $\hat\eta_x$ has maximal rank on a neighborhood of v, even if v is a focal normal of horizontal type. This means there is a neighborhood U of v in νM_x such that $\hat\eta_x|U : U \to G_k(TN)$ is an embedding into $G_k(TN)$ and such that the footpoint set V of $\hat V := \hat\eta_x(U)$ is contained in $\bar B_{\varepsilon'}(p)$. We take a ball neighborhood P of x in M_x and a neighborhood U_0 of v in $\nu_x M_x$ such that $\phi : P \times U_0 \to U; (y, w) \to w_y$ is an injective immersion into U, where w_y is the normal parallel displacement of w to y. We reduce U to the image of ϕ so that ϕ becomes a diffeomorphism onto U. We choose an f-regular point p' in $B^{\Sigma_x}_{\varepsilon'}(p)$, such that $p \in B^{\Sigma_x}_{\varepsilon}(p')$ for some ε with $0 < \varepsilon < \varepsilon'$. The map $\eta_x|\phi(\{y\} \times U_0)$ is a diffeomorphism onto its image V_y for any $y \in L$ by choice of U (note that U does not contain any focal normals of vertical type). We shrink U_0 such that this map is a diffeomorphism onto $V_x = B^{\Sigma_x}_{\varepsilon}(p')$ for $y = x$.

Lemma 3.10. *The map $\alpha_y : V_x \to V_y; \eta_x(v_x) \mapsto \eta_x(v_y)$ is an isometry, where $v_x \in U_0$ and v_y is the normal parallel displacement of v_x to $y \in P$.*

Proof. The set $V_r = V \cap N_r$ is open and dense in V and $U_r = \eta_x^{-1}(V_r)$, saturated by leaves of the shape $P \times \{w\}, w \in U_0$, is open and dense in U. We consider the diffeomorphism $\eta : U_r \to V_r$. The bifoliation on U_r is mapped to the bifoliation $(\mathcal{F}_r, \mathcal{F}_r^\perp)$ restricted to V_r. A normal foliated field on U_r maps to a normal foliated field on V_r. This is a parallel normal field when restricted to the plaques of $\mathcal{F}_r|V_r$. Moreover, any such parallel normal field along a regular plaque is given this way. If $w \in U_0$ is f-regular, $P_w = \eta(\phi(P \times \{w\}))$ and X is a parallel normal field on P_w, then $\|(\alpha_y)_* X(\eta(w))\| = \|X(\alpha_y(\eta(w)))\| = \|X(\eta(w))\|$. It follows that $\alpha_y : V_x \cap N_r \to V_y$ is a local isometry. As $V_x \cap N_r$ is open and dense in V_x, $\alpha_y : V_x \to V_y$ is an isometry. □

There is exactly one $v' \in U_0$ with $\eta(v') = p'$. Let $P' := \eta_x(\phi(P \times \{v'\}))$. We define the diffeomorphism $h : P \to P'; y \mapsto \eta_x(\phi(y, v'))$. Similarly as for U we have a natural diffeomorphism $\phi' : P' \times U'_0 \to B_\varepsilon(\nu P')$. By the splitting of η_x, $\hat\eta_x(\phi(y, v')) = \nu_{h(y)} P'$. The map $\eta_{p'} : B_\varepsilon(\nu_q P') \to B^{\Sigma_q}_\varepsilon(q)$ is a diffeomorphism for any $q \in P'$ by the choice of ε and $B^{\Sigma_{h(y)}}_\varepsilon(h(y)) = V_y$ for any $y \in P$. Then $\hat\eta_{p'} \circ (\phi'(\{h(y)\} \times U'_0))$ is equal to the transversal plaque $\hat V_y$ for any $y \in P$ $(*)$. Moreover, the map $k : U_0 \to U'_0$ defined by $k(w) = (\eta_{p'}|\phi'(\{p'\} \times U'_0))^{-1}(\eta_x(\phi(x, w)))$ is a diffeomorphism. Now let $w \in U_0$ be an arbitrary f-regular vector and $u = k(w) \in U'_0$. We extend w and u to parallel normal fields on P respectively P'. The images of $\eta_x \circ w$ and $\eta_{p'} \circ u$ lie in the same plaque of $\mathcal{F}_r|V_r$ by the splitting of η_x and $\eta_{p'}$. As $\hat\pi$ is injective on $\hat\pi^{-1}(N_r)$, the image of $\hat\eta_{p'} \circ u$ lies in the plaque $\hat\eta_x(\phi(P \times \{w\}))$ in $\hat V$. Together with $(*)$ we have $\hat\eta \circ w = \hat\eta_{p'} \circ u \circ h$ on P. By continuity we have

$$\hat\eta_x \circ \phi(y, w) = \hat\eta_{p'} \circ \phi(h(y), k(w))$$

for any $y \in P$ and $w \in U_0$.

So far we have the following. Given any k-plane $\xi \in \hat N$, any normal vector v of a parallel manifold M_x (where x is the footpoint of v) with $\hat\eta_x(v) = \xi$, that is not a focal normal of vertical type, defines as above a neighborhood $\hat V$ of ξ. A chart is given by $\hat\eta_x : U \to \hat V$. The discussion above implies that any two chart domains V intersect in open subsets of each other. So the union of topologies on the various neighborhoods V forms a basis for the topology on $\hat N$, and $\hat N$ is a topological manifold. In addition we see that the change of coordinates (h, k) is differentiable, so $\hat N$ carries a differentiable structure. Since $\hat\eta_x$ is also differentiable

as a map into $G_k(TN)$, the differentiable structure is the unique one for which the inclusion $\hat{N} \to G_k(TN)$ is an immersion. Moreover, the chart $\hat{\eta}_x : U \to \hat{V}$ induces two foliations on $\hat{V}$ that are complementary to each other. The leaves of the first are given by $\hat{\eta}_x(\phi(P \times \{*\}))$, the second by $\hat{\eta}_x(\phi(\{*\} \times U_0))$. A look at the change of coordinates (h, k) reveals that these local foliations coincide on intersections. This gives us a (vertical) foliation $\hat{\mathcal{F}}$ and a complementary (horizontal) foliation $\hat{\mathcal{F}}^\perp$ on $\hat{N}$. We will state this result in the next proposition.

Since we have not yet defined a metric on $\hat{N}$, the denotation of $\hat{\mathcal{F}}^\perp$ has to be justified. The Grassmann bundle carries a canonical metric (see appendix in [Tö]) for which the projection $G_k(TN) \to N$ is a Riemannian submersion, and the horizontal distribution of this bundle is given as follows. Let $\xi \in G_k(TN)$ be a k-plane through a point $p \in N$ spanned by an orthonormal k-frame $(v_1, \ldots, v_k)$. Then the horizontal lift $\tilde{c}$ of a curve c in N with $c(0) = p$ to ξ is given by

$$\tilde{c}(t) = \operatorname{span}\Big\{ (\overset{t}{\underset{0}{\|}} c) v_1, \ldots, (\overset{t}{\underset{0}{\|}} c) v_k \Big\}.$$

In particular, the tangent bundle $T\Sigma$ of a totally geodesic submanifold Σ of N is horizontal with respect to the projection $G_k(TN) \to N$. We denote the pullback of this metric under ι by $\hat{g}$. Note that $\hat{\pi}|T\Sigma : T\Sigma \to \Sigma$ is then an isometry.

Proposition 3.11. *$\hat{N}$ carries a natural differentiable structure, for which the inclusion into $G_k(TN)$ is an immersion. Moreover $\hat{N}$ has a natural Riemannian/totally geodesic bifoliation $(\hat{\mathcal{F}}, \hat{\mathcal{F}}^\perp)$ with respect to the pullback metric $\hat{g}$ of $\hat{N}$ in $G_k(TN)$. We have*

$$\hat{\mathcal{F}}^\perp = \{T\Sigma \mid \Sigma \text{ is a section of } M\}.$$

This proposition is a strengthening of Boualem's result in [Bou]. He states it for some differentiable structure and some metric on $\hat{N}$. We prove it for the natural differentiable structure and metric $\hat{g}$. Moreover we do not need that the leaves are relatively compact. We will call $(\hat{N}, \hat{\mathcal{F}}, \hat{\mathcal{F}}^\perp)$ the *blow-up* of $(N, \mathcal{F})$, when we have established that $\mathcal{F}$ is a singular Riemannian foliation.

Proof. The statements about the differentiable structure and the existence of the bifoliation were derived above. The description of $\hat{\mathcal{F}}^\perp$ is clear. We only have to show orthogonality. Then, since the leaves of $\hat{\mathcal{F}}^\perp$ are totally geodesic, the duality implies that $\hat{\mathcal{F}}$ is a Riemannian foliation. We consider a chart $\hat{\eta}_x : U \to \hat{V}$. For $v \in U$ with footpoint x and a horizontal vector $X \in T_vU$ and a vertical vector $Y \in T_vU$. We have

$$\hat{g}\big(d\hat{\eta}(v)X, d\hat{\eta}(v)Y\big) = g\big(d\eta(v)X, d\eta(v)Y\big) = 0.$$

The first equality is valid because $d\hat{\eta}(v)Y \in T_{\hat{\eta}(v)}T\Sigma$ is horizontal for $\pi : G_k(TN) \to N$, $\hat{\pi} \circ \hat{\eta} = \eta$ and because π is a Riemannian submersion. The second equality follows from $d\eta(v)Y \in T_{\eta(v)}\Sigma_x$ and $d\eta(v)X \perp T_{\eta(v)}\Sigma_x$ by the splitting of η. This implies that $\hat{\mathcal{F}}^\perp$ is the orthogonal foliation to $\hat{\mathcal{F}}$ with respect to $\hat{g}$. $\square$

Definition 3.12. Let $\mathcal{F}_i$ be a partition of N_i for $i = 1, 2$ into injectively immersed submanifolds. A map $f : (N_1, \mathcal{F}_1) \to (N_2, \mathcal{F}_2)$ is *foliated*, if it maps each element of $\mathcal{F}_1$ onto an element of $\mathcal{F}_2$.

From the discussion before Proposition 3.11 we have the following:

Lemma 3.13. *For an f-regular point x the map $\hat{\eta}_x : \nu M_x \to \hat{N}$ is foliated with respect to the natural bifoliation $(\mathcal{P}, \mathcal{P}^\perp)$ on νM_x and $(\hat{\mathcal{F}}, \hat{\mathcal{F}}^\perp)$ on $\hat{N}$.* □

Proposition 3.14 (Stability)**.** *If M has parallel focal structure, so has every parallel manifold M_v. If M is properly immersed (therefore embedded) and has finite normal holonomy, so does every parallel manifold.*

Proof. Let L be a leaf of $\hat{\mathcal{F}}$ in $\hat{N}$. We claim that $c(L) := \ker d(\hat{\pi}|L)(X)$ does not depend on the choice of $X \in L$. Let x be f-regular and $v \in \nu \bar{M}_x$ with $\hat{\eta}(v) \in L$. Then the image of $\hat{\eta}_x \circ v$ is L, because $\hat{\eta}_x$ is foliated. We know that $\hat{\eta}_x \circ \bar{v}$ has maximal rank. We consider the formula $\eta_x \circ \bar{v} = (\hat{\pi}|L) \circ \hat{\eta}_x \circ \bar{v}$. For $x \in M$, the map $\eta_x \circ \bar{v} = \eta \circ \bar{v}$ has constant rank by assumption, so $c(L)$ is independent of $X \in L$ and equal to the horizontal multiplicity of v. Conversely, the formula now implies that $\eta_x \circ \bar{v}$ has constant rank for an arbitrary f-regular point x and $v \in \nu_x M_x$ with $\hat{\eta}_x(v) \in L$. It follows that M_x has parallel focal structure.

Since $\eta|\bar{B}_r(\nu M)$ is proper for any $r \geq 0$ and M has finite normal holonomy, each parallel manifold is properly immersed. Being leaves (Proposition 3.8) they are embedded. It remains to show that every parallel manifold of M has finite normal holonomy. Let M_x be the parallel submanifold through a point x and Γ_x be the normal holonomy group of M_x in x, acting on $\nu_x M_x$. Since the parallel manifolds are closed and embedded, the orbits of Γ_x are discrete and compact, thus finite. By linearity Γ_x is finite. □

Ewert states this result in Proposition 2.9 in [Ew], but his proof is not correct. In the fourth last line of p. 20 he writes that $V_* \partial_t(1, \cdot, t)$ is a parallel normal field along the focal submanifold through $V(1, 0, t)$. This is not true. He refers to Proposition 2.4, [Ew], which is not correct if M_z is a focal submanifold; take $x := z \circ c$ for instance.

We define $\hat{M}_x = \hat{\pi}^{-1}(M_x)$ for f-regular $x \in N$. Note that $\hat{\pi} : \hat{\pi}^{-1}(N_r) \to N_r$ is a foliated isomorphism. If we already knew that $\mathcal{F} = \{M_v \mid v \in \nu M\}$ is a global foliation we would have that $\hat{\pi} : (\hat{N}, \hat{\mathcal{F}}) \to (N, \mathcal{F})$ is foliated. Later we prove that $\hat{M}_x$ is connected also for the endpoint x of a focal normal of horizontal type. This will show that $\mathcal{F}$ is a global foliation.

3.3. **Global Foliation.** We know that the parallel submanifolds are injectively immersed and orthogonal to the sections in each point of intersection. So far this is not clear for the focal submanifolds. This means that neither $T_p M_v$ nor $\nu_p M_v$ is defined for $p \in M_v \subset N$. Let $v \in \nu_x \bar{M}$ be a focal normal of horizontal type and $p = \eta(v)$. Let $F = F_{\bar{v}_x}$ be the focal leaf associated to v containing x. Define $F'_v = \bar{v}(F)$ and $W = (d(\eta \circ \bar{v})(x)(T_x \bar{M}))^\perp$. Up to this point we have not assumed properness of φ.

Lemma 3.15. *Let $\varphi : M \to N$ be a proper immersion with parallel focal structure and finite normal holonomy. Let $v \in \nu_x \bar{M}$ be a focal normal of horizontal type, $p = \eta(v)$ and x f-regular. Then $\bigcup_{y \in F} \hat{\eta}_x(\bar{v}_y) = W$.*

Proof. First we prove the inclusion from left to right. The rank theorem states that we can write $\eta \circ \bar{v} : \bar{M} \to N$ locally in coordinates as $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-\mu(v)}, 0, \ldots, 0)$, where $\mu(v)$ is the horizontal multiplicity of v. Since the focal leaf F is compact, we find a neighborhood U of F that is saturated by focal leaves such that $\eta \circ \bar{v} : U \to P$ is a fibration onto its image $P \subset M_v$. In

particular $(d(\eta\circ\bar v)(y)(T_y\bar M))^\perp = W$ for every $y\in F$. The splitting of η implies $A_y := \hat\eta(\bar v_y)\subset W$ for every $y\in F$.

Now let $w\in W$ be arbitrary. We look for a $y\in F$ such that $w\in A_y$. $F' = \bar v(F)$ is compact since $\eta|\bar B_r(\nu M)$ is proper for any $r\geq 0$ and φ has finite normal holonomy. The time one map ϕ^1 of the geodesic flow maps F' diffeomorphically onto a compact submanifold F^1 of W. Therefore we find a shortest ray γ in W from F^1 to w. Then γ is orthogonal to F^1 in some point $v' := \phi^1(\bar v_y)$, $y\in F$. As we will soon see $T_{v'}(A_y) = \nu_{v'}F^1$ in W (we have $A_y\subset W$), which implies that γ and therefore w lies in A_y. We want to show $T_{v'}(A_y) = \nu_{v'}F^1$. First we prove $T_{v'}(A_y)\subset\nu_{v'}F^1$. We have

$$T_{v'}F^1 = \{(0, J'_\xi(1)) \mid \xi\in T_{\bar v_y}F'\},$$

where, because $T_{v'}F^1$ consists of elements $d\phi^1(\bar v_y)\xi = (J_\xi(1), J'_\xi(1)) = (0, J'_\xi(1))$ for $\xi\in T_{\bar v_y}F'$. This implies $T_{v'}F^1\subset T_{v'}T_pN = V^N_{v'}$. Since ξ is horizontal, $J'_\xi(t)$ is orthogonal to $T_{\gamma_{\bar v_y}(t)}\Sigma_y = A_y$ by the splitting of η. So $T_{v'}(A_y)\perp T_{v'}F^1$ also in $T_{v'}T_pN = V^N_{v'}$ for the Sasaki metric, hence $T_{v'}(A_y)\subset\nu_{v'}F^1$, where we consider F^1 as a submanifold of W. Since $\dim F^1 = \dim F = \mu(v)$ and $\dim W = \mu(v)+k$, where $\mu(v)$ is the horizontal multiplicity of v and k the codimension of M, we have $T_{v'}(A_y) = \nu_{v'}F^1$ by equality of dimensions. □

Let x be f-regular, $\eta = \eta_x, M = M_x$.

Lemma 3.16. *Let $v\in\nu_x\bar M$ be a focal normal of horizontal type that is not of vertical type. Then there is a neighborhood O of x that is saturated by focal leaves of v, a relatively compact set P, $\varepsilon>0$, a neighborhood U_0 of $\bar v_x\in\nu_x\bar M$ such that*

1. *$\eta\circ\bar v : O\to P$ is a surjective trivial fibration whose fibers are the focal leaves, i.e. the trivialization has the shape $O\cong F_v\times P$.*
2. *The map $\tilde\eta : O\times U_0\to T(P,\varepsilon); (y,\bar w_x)\mapsto\eta(\bar w_y)$ is onto the tube $T(P,\varepsilon)$.*
3. *$F_{\bar w_y}\subset F_{\bar v_y}$ for any $(y,\bar w_x)\in O\times U_0$. This means that the focal foliation given by $\eta\circ\bar w$ is finer than the focal foliation given by $\eta\circ\bar v$.*
4. *Each section through a point $q\in T$ also contains the unique point p' in P that is in the same slice as q.*
5. *Let $p\in P$ and S_p be the slice in T through p. Then $S_q\subset S_p$ for any $q\in S_p$.*

Proof. Any normal parallel translation v' of v with footpoint y has the same multiplicity for $\exp^{\Sigma_y}$ as v for $\exp^{\Sigma_x}$ as a consequence of Lemma 3.10. Thus v' is not a focal normal of vertical type. As before there is a neighborhood O of the focal leaf F saturated by focal leaves and a neighborhood U_0 of v in $\nu_x\bar M$ such that $\eta\circ\bar v|O$ is a fibration onto its image P, which we can assume to be relatively compact, with typical fiber F and such that $\hat\eta\circ\phi : O\times U_0\to\hat N$ is a diffeomorphism onto its image, where $\phi : O\times U_0\to\nu M; (y,w)\mapsto\bar w_y$. In particular $\hat\eta\circ\phi|(\{y\}\times U_0)$ is a chart for $\hat{\mathcal F}$. Let $\varepsilon>0$ be smaller than the injectivity radius $i_N(q)$ in N for all $q\in P$. Shrinking U_0 we can then assume that $\eta\circ\phi|(\{y\}\times U_0)$ is a diffeomorphism onto its image V_y for any $y\in F$. We can also assume that V_x is the image of the ε-ball in $\hat\eta(\bar v_x)\subset T_pN$ around the origin under exp (if Σ had no self-intersections we would write $V_x = B^{\Sigma_x}_\varepsilon(p)$). By Lemma 3.10, $\alpha_y : V_x\to V_y$ is an isometry. As $\eta\circ\phi(y,\,\cdot\,) = \alpha_y\circ\tilde\eta\circ\phi(x,\,\cdot\,)$ we have that $\eta\circ\phi(\{y\}\times U_0) = V_y$ is the image of the ε-ball in $\hat\eta(\bar v_y)\subset T_pN$ around the origin under exp. Then $\eta\circ\phi : O\times U_0\to T$ is surjective because the slice S_q of P in T through $q\in P$ is equal to

$$S_q = \bigcup\{V_y \mid y \text{ is in the focal leaf associated to } v \text{ through } y\}$$

for any $q \in P$ by Lemma 3.15.

Let $y \in O$ and $u \in U_0$ be arbitrary. Let $F' = F_{\bar{u}_y}$ be the focal leaf associated to u through y. Let $q = \eta(\bar{u}_y)$ and $p' = \eta(\bar{v}_y) \in P$. We want to show that F' is contained in the focal leaf F associated to v through y. This is clear if u is f-regular. We assume that u is a focal normal of horizontal type. Obviously V_y contains p' and q. There is a vector $w \in \hat{\eta}(\bar{v}_y) \subset \nu_{p'}P$ of length smaller than ε with endpoint q. For $z \in U$ we define $w_z = d\alpha_z(p')w$, where $\alpha_z : V_y \to V_z$ as above but with central point p' instead of p. The endpoint $\alpha_z(q)$ of w_z is still in $T(P, \varepsilon)$ because $\|w_z\| = \|w\| < \varepsilon$ for all $z \in O$. For all $z \in F' \subset O$ we have $q = \eta(\bar{u}_z) = \alpha_z(q)$, thus $w_z = w$ since w is unique among the vectors of νP of length smaller than ε with endpoint $q \in T$. Therefore $\eta(\bar{v}_z) = \alpha_z(p') = p'$ for all $z \in F'$, so $F' \subset F_{\bar{v}_y}$. (In other words, the foliation of focal leaves given by $\eta \circ \bar{u}$ is finer than the foliation of focal leaves given by $\eta \circ \bar{v}$.) Therefore

$$S_q \subset \bigcup_{y \in F'} V_y \subset \bigcup_{y \in F} V_y = S_{p'}.$$

□

We can deduce that the set $\hat{\pi}^{-1}(p)$ of tangential spaces in p of sections through p is equal to $\hat{\eta}(F'_v)$, where $\eta(v) = p$. To see this let $X \in \hat{\pi}^{-1}(p)$. Let Y be the image of a small ball in X around the origin, such that $Y \subset T(P, \varepsilon)$. Assume that X is not contained in $W = \bigcup_{y \in F} \hat{\eta}(\bar{v}_y)$. Then there is an f-regular point $z \in Y$ that lies in a slice S_q for $q \in P \backslash \{p\}$. Since there is only one section through an f-regular point, Y has to lie in S_q and thus cannot contain p, contradiction. Now we will show in the following theorem that M_v is embedded with the help of the blow-up.

Compare the first of following statements with the weaker result of Corollary 2.14 in [Ew]. That corollary is based on Lemma 2.13, [Ew] which is not proved correctly (see the first sentence of the proof).

Theorem 3.17. *If $\varphi : M \to N$ is a proper immersion with parallel focal structure and finite normal holonomy, then $\mathcal{F} = \{M_v \mid v \in \nu M\}$ is a transnormal global foliation and the leaves of $\mathcal{F}$ are closed, embedded and orthogonal to each section they meet. Moreover, $\mathcal{F}$ is a singular Riemannian foliation admitting sections.*

Proof. Assume $\eta(v) = \eta(w) =: p$ for $v, w \in \nu \bar{M}$. We have to show $M_v = M_w$. If p is f-regular, then v and w are tangential to the same section by Lemma 3.6, so $\hat{\eta}(v) = \hat{\eta}(w)$. By Proposition 3.13 it follows $\hat{\eta} \circ \bar{v}(\bar{M}) = \hat{\eta} \circ \bar{w}(\bar{M})$ and therefore $M_v = M_w$ because $\hat{\pi} \circ \hat{\eta} = \eta$. Now let p be a focal point of horizontal type. It is possible that $\hat{\eta}(v) \neq \hat{\eta}(w)$. Both elements lie in $\hat{\pi}^{-1}(p) = \bigcup_{y \in F} \hat{\eta}(\bar{v}_y)$. Therefore there is a normal parallel translation v' of v with $\hat{\eta}(v') = \hat{\eta}(w)$. As above we conclude $M_v = M_{v'} = M_w$, so $\mathcal{F}$ is a global foliation.

We already know that the parallel submanifolds are closed and embedded by Proposition 3.14. Now we consider M_v, where v is a focal normal of horizontal type, and $p \in M_v$ arbitrary. Assume $\eta(v) = \eta(w)$ for $v, w \in \nu \bar{M}$ with footpoint x, y. Then by Lemma 3.16 we find neighborhoods O_1, O_2 of x, y that are saturated with focal leaves for $\eta \circ \bar{v}$ respectively $\eta \circ \bar{w}$ such that $\hat{\eta} \circ \bar{v}|O_1$ and $\eta \circ \bar{w}|O_2$ have the same image. Then $\eta \circ \bar{v}$ and $\eta \circ \bar{w}$ have the same image. As $\eta \circ \bar{v} : \bar{M} \to N$ is proper, M_v is closed and embedded. Now νM_v is well-defined for any $p \in M_v$. The previous discussion showed that $\nu_p M_v$ is the union of all $T_p\Sigma$, where Σ is a

section through p. This implies that also a focal submanifold intersects each section it meets orthogonally. This also implies that $\mathcal{F}$ is transnormal.

To prove that $\mathcal{F}$ is a singular Riemannian foliation, it remains to show that $\Xi(\mathcal{F})$ acts transitively at a given point p. This is clear for f-regular $p \in N$, since the set of f-regular points N_r is foliated by $\mathcal{F}_r$. Therefore we can assume that p is a focal point of horizontal type of M and $v \in \nu_x \bar{M}$ with $\eta(v) = p$. We assume that v is not a focal normal of vertical type, otherwise we replace M by a parallel manifold. Now we use the same objects as in Lemma 3.16. We want to define a distribution $\mathcal{D}'$ of dimension $\dim P$ on $T = T(P, \varepsilon)$ such that $\mathcal{D}'(q) \subset T_q M_q$. Let $q \in T$ be arbitrary. Let S_q be a slice of M_q through q. Then there is a unique point $p' \in P$ such that the slice $S_{p'}$ of T through p' contains q. We define $\mathcal{D}'(q) = T_q S_{p'}^{\perp}$. Since the distribution tangential to the slices is differentiable so is $\mathcal{D}'$. Since $S_q \subset S_{p'}$ by Lemma 3.16, we have $\mathcal{D}'(q) \subset T_q M_q$. Thus, for any $p \in P$ and $X_0 \in T_p P$ there is a vector field X of $\mathcal{D}'$ in T extending X_0. If $f : N \to \mathbb{R}$ is a bump function with support in U and $f(p) = 1$ then $fX \in \Xi(\mathcal{F})$ with $(fX)_p = X_0$. Since p and X_0 were arbitrary, $\Xi(\mathcal{F})$ acts transitively on $T\mathcal{F}$. $\square$

We can now exploit the theory of singular Riemannian foliations for submanifolds with parallel focal structure. Implications will be given in the next section. The converse was proven in [A]. We give a different proof in the next section, see Theorem 4.3.

Remark 3.18. Due to Lemma 3.13 $\hat{\eta} : (\nu M, \mathcal{P}, \mathcal{P}^{\perp}) \to (\hat{N}, \hat{\mathcal{F}}, \hat{\mathcal{F}}^{\perp})$ is foliated. Since $\mathcal{F}$ is a global foliation, $\eta : (\nu M, \mathcal{P}) \to (N, \mathcal{F})$ is foliated, and then, because of $\eta = \hat{\pi} \circ \hat{\eta}$, also $\hat{\pi} : (\hat{N}, \hat{\mathcal{F}}) \to (N, \mathcal{F})$. Each parallel submanifold has the same focal submanifolds by the theorem.

A focal point of horizontal type of M is also a focal point with the same horizontal multiplicity of any other parallel submanifold of M and vice versa. For a focal normal v of horizontal type $\eta \circ v : \bar{M} \to M_v$ is locally trivial fibration by Lemma 3.16, and so is the restriction of $\hat{\pi}$ to $\hat{M}_p$ for a focal point p of horizontal type.

The starting point of our work was the question, under which conditions a submanifold M in N with minimal conditions (1) and (2) stated in the introduction induces a global foliation $\mathcal{F}$ by parallel and focal submanifolds. For such a submanifold M that has in addition sections Σ_x for every $x \in M$ a necessary and sufficient condition to induce a global foliation is that M admits sections. Sufficiency is provided by Theorem 3.17. Necessity is clear. Otherwise there is a regular point p and two sections Σ_1 and Σ_2 with $T_p\Sigma_1 \neq T_p\Sigma_2$. Then there are two parallel manifolds M_{v_i} with $T_p M_{v_i} \perp T_p \Sigma_i$ $(i = 1, 2)$, thus $T_p M_{v_1} \neq T_p M_{v_2}$, contradicting that M induces a global foliation.

We call the elements of $\mathcal{F}$ *leaves*. A leaf is called *regular* if its dimension is maximal in $\mathcal{F}$, otherwise *singular*. A regular leaf with non-trivial normal holonomy is called *exceptional*.

A point in N is f-regular if and only if it is contained in a regular leaf of $\mathcal{F}$. This justifies the denotation: the f in f-regular stands for *foliation*.

4. Singular Riemannian Foliations

4.1. **Parallel Focal Structure of Regular Leaves.** Let $\mathcal{F}$ be a singular Riemannian foliation admitting sections of a complete Riemannian manifold N. (For an introduction to singular Riemannian foliations, see section 6 in [Mo]) Let M be

a regular leaf and $\eta : \nu M \to N$ be its normal exponential map. We recall that a foliated vector field normal to the foliation on a simple neighborhood of a regular point (for $\mathcal{F}$) are parallel normal fields when restricted to the (regular) leaves. We can derive that νM is flat. Therefore νM is endowed with a natural foliation of horizontal leaves. The existence of sections implies the splitting of η. Therefore we can speak of f-regular points and vectors and of focal normals of horizontal/vertical type. From [Mo] we know that the stratum of regular points is open and dense in N. The following lemma is not difficult to prove. The second statement follows from the first with Corollary 3.7.

Lemma 4.1 ([Tö]). *A point of N is $\mathcal{F}$-regular if and only if it is f-regular. In particular the subset of $\mathcal{F}$-regular points in a section is open and dense.* □

Hence a regular leaf M admits sections in the sense of section 3.

Proposition 4.2. *The map $\eta : \nu M \to N$ is foliated and the restriction of η to a horizontal leaf in νM has constant rank.*

Proof. Let $v \in \nu M$ with endpoint p and footpoint x. We define

$$Z = \{w \in L_v \mid \eta(w) \in M_p\},$$

where L_v is the horizontal leaf of νM through v. We want to show that Z is open and closed in L_v and therefore equal to L_v by connectivity. Let $w \in Z$ with footpoint y and $q = \eta(w)$. Let P_q be a relatively compact open neighborhood of q in M_q and let T be an injectivity tube around P_y (which is a distinguished neighborhood of P_q in the sense of 6.2 in [Mo]). Since $\Xi(\mathcal{F})$ acts transitively on $T\mathcal{F}$ we can assume that each plaque in T intersects each slice of T and always transversally. Thus the restriction of the projection $\rho : T \to P_q$ to an arbitrary plaque in T is a surjective submersion. We choose a positive number $t < 1$ such that tv is f-regular and $\gamma_w|[t,1]$ lies in T. We see that γ_w intersects P_q orthogonally for $t = 1$ since $\mathcal{F}$ is transnormal and $\gamma_w|[t,1]$ lies in the slice of T through q. The leaf M_{tv} is regular by the previous lemma. Let $L' = L_{(1-t)\phi^t(v)}$ be the horizontal leaf in νM_{tv} containing $(1-t)\phi^t(v)$. Observe that the map $\alpha : L_v \to L'; \xi \mapsto (1-t)\phi^t(\xi)$ is a diffeomorphism and that $\eta_x|L_v = (\eta_{\eta(tv)} \circ \alpha)|L_v$. This means that we can replace M by M_{tv} for our considerations and assume that $\gamma_w|[0,1]$ is contained in a slice of T, so in particular the footpoint y of w lies in T. Let P_y be the connected component of M_y in T containing y. We define the function $r : T \backslash P_q \to \mathbb{R}$ measuring the distance to P_q and let $X = -\operatorname{grad} r$ be the negative of the radial vector field. Then $w = \|w\| X_y$. Note that $X|P_y$ is a normal vector field of P_y. The flow of X is a family of homotheties in T centered at P_q which respects the singular Riemannian foliation by the Homothety Lemma (see Lemma 6.2 in [Mo]). Due to Proposition 2.2 in [Mo] X is a foliated vector field on a neighborhood of P_y in T. Thus $X|P_y$ is a normal parallel field of P_y and the image of P_y under X is an open subset of the horizontal leaf L_v in νM_x containing w. We want to show that $(\eta \circ (\|w\|X))|P_y = \rho|P_y$ which implies that Z is open in L_v. But this follows from the observation that $\phi_X(t,z) = \gamma_{X_z}(t)$ for $t \in [0, \|w\|)$ and $z \in P_x$ where ϕ_X is the flow of X; note that $\|w\|$ is the distance of P_y and P_q. We remark that this implies that $\eta|L_v$ has constant rank and its image is open in M_p. Now let $w \notin Z$ with footpoint y and endpoint q. By assumption $q \notin M_p$. As above we show that an open neighborhood of w in L_v is mapped to M_q which is disjoint to M_p by definition of $\mathcal{F}$. Therefore the complement of Z is also open. Thus $\eta(L_v) \subset M_p$.

We will now show $\eta(L_v) = M_p$. We have seen above that $\eta(L_v)$ is open in M_p. It suffices to show that $\eta(L_v)$ is also closed in M_p. Let q be an arbitrary point on the boundary of $\eta(L_v)$ in M_p. We have to show $q \in \eta(L_v)$. There is an injectivity tube T of some open neighborhood P_q of q in M_q. As $\Xi(\mathcal{F})$ acts transitively on $T\mathcal{F}$, we can assume that any plaque in T meets any slice of P_q, and always transversally. Now there is a $w \in L_v$ such that $\eta(w) \in P_q$. As above we can assume that the footpoint y of w is contained in T. Then we define $X = -\operatorname{grad} r$ on $T \backslash P_q$ and we have $w = \|w\| X_y$. The endpoint of $\|w\| X_{y'}$ for $y' \in P_y$ is the unique point in the intersection of P_q and the slice of P_q containing y'. Since $P_{y'}$ meets any slice of P_q, in particular the slice through q, we have $q \in \eta(L_v)$ and $\eta(L_v) = M_p$. □

As a direct consequence of the lemma we obtain the following theorem of M. Alexandrino.

Theorem 4.3 ([A])**.** *A regular leaf of a singular Riemannian foliation admitting sections of a complete Riemannian manifold has parallel focal structure.* □

Remark 4.4. Let M be a regular leaf and $v \in \nu M$ with footpoint x and singular endpoint p. We want to see that there is a compact submanifold $F \subset M$ of dimension equal to the horizontal multiplicity of v to which we can extend v to a parallel normal field such that image of $\eta \circ v$ is p. As above we can assume that x is contained in an injectivity tube T of a relatively compact singular plaque P_p and that $\gamma_v : [0,1] \to N$ is the minimal normal geodesic from x to p. Let F be a connected component of the intersection of a regular leaf M with the slice through p containing x. As in the proof of Lemma 4.2, v can be extended to a parallel normal field on a neighborhood of F which coincides with the restriction of a radial field up to a scalar. The image of $\eta \circ v|F$ is p and the corresponding focal leaf in $\bar{M}$ covers F. We call F focal leaf in M.

The following is a slice theorem for singular Riemannian foliations admitting sections.

Theorem 4.5 ([A])**.** *Let $\mathcal{F}$ be a singular Riemannian foliation admitting sections of a complete Riemannian manifold N. Let $p \in N$, $B_\varepsilon(0_p)$ be the ball of 0_p in $\nu_p M_p$ for a small radius ε and $S_p = \exp^\perp(B_\varepsilon(0_p))$. Then the restriction $\mathcal{F}|S_p$ is a singular Riemannian foliation admitting sections that is isomorphic to the restriction of an isoparametric partition $\mathcal{F}'$ of $\mathbb{R}^m$ to a ball neighborhood of the origin with the same codimension. This isomorphism is given by $\exp^\perp : B_\varepsilon(0_p) \to S_p$, and it maps flat sections of $\mathcal{F}'$ to sections of $\mathcal{F}$ restricted to S_p.* □

An isoparametric partition is invariant under homotheties, i.e. maps $h_\lambda : \mathbb{R}^m \to \mathbb{R}^m; x \mapsto \lambda x$, where $\lambda \in \mathbb{R}$. Therefore one can recover the isoparametric partition from its restriction to an open neighborhood of the origin. An isoparametric family of submanifolds of $\mathbb{R}^m$ is given as the level sets of a transnormal map. Therefore the isoparametric family in $\mathbb{R}^m$ and $\mathcal{F}|S_q$ are proper singular Riemannian foliations, i.e., its leaves are closed and embedded.

4.2. **Transversal Holonomy.** Let $(N, \mathcal{F})$ be as in the previous section. Then by Theorem 4.3 and section 3

$$\hat{N} := \{T_p\Sigma \mid p \in N, \Sigma \text{ is a section of } \mathcal{F} \text{ through } p\}$$

carries the unique differentiable structure for which the inclusion $\hat{N} \to G_k(TN)$ is an immersion (see Proposition 3.11). Moreover, $\hat{N}$, endowed with the pull-back metric, carries by a Riemannian/totally geodesic bifoliation $(\hat{\mathcal{F}}, \hat{\mathcal{F}}^\perp)$, where

$$\hat{\mathcal{F}}^\perp = \{T\Sigma \mid \Sigma \text{ is a section of } \mathcal{F}\}.$$

We call $\hat{\mathcal{F}}$ *vertical* and $\hat{\mathcal{F}}^\perp$ *horizontal* foliation. Since $\hat{\eta}$ is foliated by Lemma 3.13 and η by Lemma 4.2, so is the footpoint map $\hat{\pi} : (\hat{N}, \hat{\mathcal{F}}) \to (N, \mathcal{F})$ because of $\eta = \hat{\pi} \circ \hat{\eta}$. The same arguments for $\mathcal{P}^\perp$ and $\hat{\mathcal{F}}^\perp$ show that $\hat{\pi}$ maps a horizontal leaf $T\Sigma$ to the section Σ. Here by Σ we mean the submanifold and not its image, which can have self-intersections in singular points. When we introduced the metric $\hat{g}$ on $\hat{N}$ in subsection 3.2, we noticed that $\hat{\pi} : T\Sigma \to \Sigma$ is an isometry.

The discussion shows that $\hat{M} = \hat{\pi}^{-1}(M)$ is a leaf of $\hat{\mathcal{F}}$ for regular M, and a union of leaves if M is singular. Let M be singular. Now we want to see that $\hat{M}$ is also connected. In subsection 3.3 we used properness and finite normal holonomy assumptions on the submanifold with parallel focal structure to prove this (see the paragraph before Theorem 3.17). If a partition is already given, in our case $\mathcal{F}$, then these conditions are not necessary. Let p in M. By definition $\hat{\pi}^{-1}(p)$ is the set of tangential spaces in p of sections through p. It suffices to show that this set is contained in one leaf of $\hat{\mathcal{F}}$. Let S_p be a slice through p. The corresponding isoparametric partition of $\nu_p M_p$ given by Theorem 4.5 has closed and embedded regular leaves with parallel focal structure and finite normal holonomy. Let L be a regular leaf of this isoparametric partition. Now Proposition 3.15 describes the set of sections through p as the image of a focal leaf associated to v under $\hat{\eta} \circ \bar{v}$ for some $v \in \nu L$, so $\hat{\pi}^{-1}(p)$ is contained in one leaf. This means $\hat{M}_p := \hat{\pi}^{-1}(M_p)$ is a leaf. Therefore

$$\hat{\mathcal{F}} = \{\hat{\pi}^{-1}(M) \mid M \in \mathcal{F}\}.$$

For a curve $\tau : [0,1] \to N$ in a regular leaf of $\mathcal{F}$ and a curve $\sigma : [0,1] \to N$ in a section, both starting in an $\mathcal{F}$-regular point, we define the *lifts* $\hat{\tau}(t) := T_{\tau(t)}\Sigma_{\tau(t)}$ and $\hat{\sigma}(t) := T_{\sigma(t)}\Sigma_{\sigma(0)}$. Obviously $\hat{\pi} \circ \hat{\tau} = \tau$ and $\hat{\pi} \circ \hat{\sigma} = \sigma$.

Lemma 4.6. *Let x_0 be $\mathcal{F}$-regular, let $\tau : [0,1] \to N$ be a curve in M_{x_0} and $\sigma : [0,1] \to N$ be a curve in Σ_{x_0} with $\tau(0) = \sigma(0) = x_0$. Then there is a unique continuous map $H = H_{(\tau,\sigma)} : [0,1] \times [0,1] \to N$ with*

1. *$H(\,\cdot\,, 0) = \tau$,*
2. *$H(0, \,\cdot\,) = \sigma$,*
3. *$H(\,\cdot\,, t)$ is contained in a leaf of $\mathcal{F}$,*
4. *$H(s, \,\cdot\,)$ is contained in a section.*

Proof. First assume that $\mathcal{F}$ is a (regular) Riemannian foliation admitting sections. The set of sections forms a totally geodesic foliation orthogonal to $\mathcal{F}$. In this case the statements are due to Corollary 2.7 of [BH1], which is based on Lemma 2.6. Note that for the proof of the latter one can drop completeness of N and assume completeness of the horizontal leaves, the sections, instead. In particular the statements are valid for $(\hat{\mathcal{F}}, \hat{\mathcal{F}}^\perp)$.

Now let $\mathcal{F}$ be a singular Riemannian foliation with sections. Existence follows by $H_{(\tau,\sigma)} := \hat{\pi} \circ H_{(\hat{\tau},\hat{\sigma})}$, where $\hat{H}_{(\hat{\tau},\hat{\sigma})}$ is defined as in the lemma for the bifoliation $(\hat{\mathcal{F}}, \hat{\mathcal{F}}^\perp)$ of $\hat{N}$. We want to show uniqueness of $H_{(\tau,\sigma)}$. Let H be arbitrary with the four properties in the lemma. We define $\hat{H}(s,t) := T_{H(s,t)}\Sigma_{\tau(s)}$. Obviously

$\hat{H}(s,\cdot)$ lies in the horizontal leaf $T\Sigma_{\tau(s)}$. The curve $H(\cdot,t) = \hat{\pi}\circ\hat{H}(\cdot,t)$ lies in the leaf $M_{\sigma(t)}$ by assumption. By the discussion at the beginning of this section, $\hat{\pi}^{-1}(M_{\sigma(t)})$ is a leaf of $\hat{\mathcal{F}}$. Therefore $\hat{H}(\cdot,t)$ is contained in a vertical leaf. By uniqueness we have $\hat{H} = \hat{H}_{(\hat{\tau},\hat{\sigma})}$ and therefore $H = \hat{\pi}\circ H_{(\hat{\tau},\hat{\sigma})}$ is also determined. □

A continuous map $H : [0,1]\times[0,1] \to N$, such that $H(\cdot,t)$ is vertical for any t and $H(s,\cdot)$ is horizontal for any s, is called *rectangle* with *initial vertical/horizontal curve* $H(\cdot,0)/H(0,\cdot)$, *terminal vertical/horizontal curve* $H(\cdot,1)/H(1,\cdot)$ and *diagonal* $t \mapsto H(t,t)$. We write $T_\sigma\tau = H_{(\tau,\sigma)}(\cdot,1)$ and $T_\tau\sigma = H_{(\tau,\sigma)}(1,\cdot)$. For a vertical curve τ in a leaf $M \in \mathcal{F}$ respectively a horizontal curve σ in a section Σ we write $[\tau]$ respectively $[\sigma]$ for the equivalence class of curves under homotopy in M respectively Σ fixing endpoints. Then $[T_\sigma\tau]$ and $[T_\tau\sigma]$ only depend on $[\tau]$ and $[\sigma]$. We remark that for any curve $\mu : [0,1] \to N$ we find a unique rectangle $H : [0,1]\times[0,1] \to N$ with diagonal μ. We write μ_v respectively μ_h for the initial vertical respectively horizontal curve of H and μ^v respectively μ^h for the terminal vertical respectively horizontal curve of H. In the sequel we make the following convention: We write $H_{(\tau,\sigma)}$ for rectangles in N and, as in the proof of Lemma 4.6, $\hat{H}_{(\hat{\tau},\hat{\sigma})}$ for rectangles in $\hat{N}$ with respect to the bifoliation $(\hat{\mathcal{F}},\hat{\mathcal{F}}^\perp)$. Then $H_{(\tau,\sigma)} = \hat{\pi}\circ\hat{H}_{(\hat{\tau},\hat{\sigma})}$ as in the proof.

We recall that the universal cover $\tilde{M}$ of a manifold M is equal to the set of equivalence classes of curves starting from a fixed point x_0, where the equivalence is given by homotopy fixing endpoints; in some cases we write more precisely $\widetilde{(M,x_0)}$. The covering map $\tilde{M} \to M$ is given by $[\sigma] \mapsto \sigma(1)$. Let x_0 be arbitrary and let $M \in \mathcal{F}$ the leaf through x_0 and Σ the section through x_0. Then

$$\begin{aligned}
\tilde{M} &= \{[\tau] \mid \tau \text{ is vertical and } \tau(0) = x_0\}\\
\tilde{\Sigma} &= \{[\sigma] \mid \sigma \text{ is horizontal and } \sigma(0) = x_0\}\\
\bar{N} &= \{[\mu] \mid \mu \text{ is a curve in } \hat{N} \text{ and } \mu(0) = x_0\},
\end{aligned}$$

where $\bar{N}$ denotes the universal covering of $\hat{N}$ and x_0 is identified with its lift $T_{x_0}\Sigma_{x_0}$. The manifold $\bar{N}$ is endowed with the pull-back bifoliation of the covering $\bar{N} \to \hat{N}$ and $\tilde{M}\times\tilde{\Sigma}$ carries a natural bifoliation. Due to [BH2] the map $\Phi : \tilde{M}\times\tilde{\Sigma} \to \bar{N}; ([\tau],[\sigma]) \mapsto [t \mapsto \hat{H}_{(\hat{\tau},\hat{\sigma})}(t,t)]$ (the diagonal) is a bifoliated diffeomorphism (i.e. foliated with respect to both pairs of foliations) with inverse map $[\mu] \mapsto ([\mu_v],[\mu_h])$ (see [Tö] for details). Consequently

$$\begin{aligned}
\Psi : \tilde{M}\times\tilde{\Sigma} &\to \hat{N}\\
([\tau],[\sigma]) &\mapsto \hat{H}_{(\hat{\tau},\hat{\sigma})}(1,1)
\end{aligned}$$

is a bifoliated universal covering map of $\hat{N}$. We define

$$\begin{aligned}
\psi : \tilde{M}\times\tilde{\Sigma} &\to N\\
([\tau],[\sigma]) &\mapsto H_{(\tau,\sigma)}(1,1).
\end{aligned}$$

By definition

$$\hat{\pi}\circ\Psi = \psi.$$

Since the footpoint map $\hat{\pi} : (\hat{N},\hat{\mathcal{F}}) \to (N,\mathcal{F})$ is foliated and maps a horizontal leaf $T\Sigma$ isometrically to the corresponding section Σ, it follows that ψ is foliated and maps horizontal leaves onto sections. We sum up:

Proposition 4.7. *The map Ψ is the universal covering map, and it is foliated with respect to the natural bifoliation of $\tilde{M}\times\tilde{\Sigma}$ and to $(\hat{N};\hat{\mathcal{F}},\hat{\mathcal{F}}^{\perp})$. The map ψ is foliated with respect to the vertical foliation on $\tilde{M}\times\tilde{\Sigma}$ and $(N,\mathcal{F})$ and its restriction to a horizontal leaf is a Riemannian covering to a section.* □

Remark 4.8. The map ψ completely describes the singular Riemannian foliation $\mathcal{F}$ of N. The singular values of ψ are exactly the singularities of $\mathcal{F}$. It is a covering when restricted to the regular set.

Let $\tau : [0,1] \to M$ be a curve with $\tau(0) = x_0$. The map $T_{\hat{\tau}} : \widetilde{(T\Sigma, x_0)} \to \widetilde{(T\Sigma, \hat{\tau}(1))}; [\sigma] \mapsto [T_{\hat{\tau}}\hat{\sigma}]$ defined with respect to $(\hat{\mathcal{F}},\hat{\mathcal{F}}^{\perp})$ is an isometry due to [BH2]. On the other hand we have $T_\tau : \widetilde{(\Sigma, x_0)} \to \widetilde{(\Sigma, \tau(1))};\ [\sigma] \mapsto [T_\tau\sigma]$ with respect to $\mathcal{F}$ and the family of sections. Since the isometry $\hat{\pi} : T\Sigma \to \Sigma$ passes to an isometry of the universal covers, we can identify $T_{\hat{\tau}}$ and T_τ. In particular:

Proposition 4.9. *The sections have the same Riemannian universal cover. Similarly the regular leaves of $\mathcal{F}$ have the same universal cover.* □

This proposition describes a topological difference between a singular Riemannian foliation admitting sections and a polar action, namely the normal holonomy of a section. While the sections of a polar action are isometric to each other, the sections of a singular Riemannian foliation only have the same Riemannian universal cover. We want to explain this in more detail. We can define a local isometry along a vertical curve τ starting in Σ similarly as in Verweisl 3.10. It is important to know that in general such a map cannot be extended to an isometry that is defined on all of Σ. For instance consider the Klein bottle $N = [0,1]^2/\sim$, where we identify the two vertical edges in opposite direction and the horizontal ones in common direction. The two partitions, the one into vertical, the other into horizontal lines, build a Riemannian/totally geodesic bifoliation, so in particular a singular Riemannian foliation admitting sections. Take M to be a vertical line and Σ to be the exceptional horizontal line. Let τ be a curve in M from a point in Σ to a point that is not in Σ. Obviously we cannot extend a local isometry defined as above to a map defined on Σ that respects the foliation. But we can develop these maps on the Riemannian universal cover $\tilde{\Sigma}$ and this is $T_{[\tau]} : \widetilde{(\Sigma, \tau(0))} \to \widetilde{(\Sigma, \tau(1))}$. The set of the above local isometries along vertical curves τ that start and end in Σ becomes a pseudogroup of local isometries on Σ while the set of its developments $T_{[\tau]}$ becomes a group acting on $\tilde{\Sigma}$ that we will later denote by $\tilde{\Gamma}$. It is more convenient to work with this group than with the corresponding pseudogroup. On the other hand we have to handle additional elements, namely the deck transformations of $\pi_\Sigma : \tilde{\Sigma} \to \Sigma$, which are contained in $\tilde{\Gamma}$, but do not contribute to the geometry of $\mathcal{F}$. For a special choice of sections we can divide them out and obtain a group Γ acting on Σ, that completely describes the holonomy of $\mathcal{F}$. In the case that $\mathcal{F}$ is the orbit decomposition of a polar action, Γ is the generalized Weyl group of a polar action.

For a vertical/horizontal curve c and a horizontal/vertical curve d (horizontal means lying in a section) starting in the same regular point x_0 we denote by $T_c d$ the terminal horizontal/vertical edge of the homotopy $H_{(c,d)}$.

$$\tilde{\Lambda} = \left\{ [\tau]T_{[\sigma]} : \tilde{M} \to \tilde{M} \,\middle|\, \begin{array}{c} \tau \text{ is vertical}, \sigma \text{ is horizontal} \\ \tau(0) = \sigma(0) = x_0, \sigma(1) = \tau(1) \end{array} \right\}$$

and

$$\tilde{\Gamma} = \left\{ [\sigma]T_{[\tau]} : \tilde{\Sigma} \to \tilde{\Sigma} \;\middle|\; \begin{array}{c} \tau \text{ is vertical}, \sigma \text{ is horizontal} \\ \tau(0) = \sigma(0) = x_0, \sigma(1) = \tau(1) \end{array} \right\}.$$

Later we focus on $\tilde{\Gamma}$. It turns out that this group (see below) carries the information about the tranversal geometry of $\mathcal{F}$. As we noted before, identifying Σ with $T\Sigma$ and their universal covers, T_τ defined with respect to $\mathcal{F}$ and the family of sections and $T_{\hat{\tau}}$ defined with respect to $(\hat{\mathcal{F}}, \hat{\mathcal{F}}^\perp)$ are the same; here it is important that we have chosen x_0 to be regular (check the assumption in Lemma 4.6). As a consequence we can identify the above two sets with their counterparts for $(\hat{\mathcal{F}}, \hat{\mathcal{F}}^\perp)$. For $\tilde{\Gamma}$ this means that the transversal geometry of $\mathcal{F}$ can be read off from that of its blow-up $\hat{\mathcal{F}}$. The rules in the next lemma work for T of $(\hat{\mathcal{F}}, \hat{\mathcal{F}}^\perp)$. For T of $\mathcal{F}$ the second rule has an exception: If c_2 is horizontal and $c_2(0)$ is singular, the assumption in Lemma 4.6 are not met and $T_{(\cdot)}c_2$ is not defined. This case will not occur in our discussion.

Lemma 4.10. *Let c_1 and c_2 both be horizontal/vertical with $c_1(0) = x_0$ and $c_1(1) = c_2(0)$ and d vertical/horizontal with $d(0) = c_1(0)$. Then*

$$T_{c_1 c_2} = T_{c_2} \circ T_{c_1} \qquad \textit{and} \qquad T_d(c_1 c_2) = T_d c_1 \cdot T_{T_{c_1} d} c_2.$$

Proof. The proof is clear. □

Lemma 4.11. *$\tilde{\Gamma}$ is a subgroup of $I(\tilde{\Sigma})$ and $\tilde{\Lambda}$ a subgroup of Diff$(\tilde{M})$.*

Proof. This is an easy application of Lemma 4.10. See [Tö]. □

We have already explained the geometric meaning of these groups. Now we want to give a relation between $\pi_1(\hat{N}, x_0), \tilde{\Gamma}$ and $\tilde{\Lambda}$. We identify the actions of $\tilde{\Gamma}$ and $\tilde{\Lambda}$ with the corresponding actions for $(\hat{\mathcal{F}}, \hat{\mathcal{F}}^\perp)$. For $[\mu] \in \pi_1(\hat{N}, x_0)$ we define $\tilde{\gamma}_{[\mu]} := [(\mu_h)]T_{[(\mu^v)^{-1}]} \in \tilde{\Gamma}$ and $\tilde{\lambda}_{[\mu]} := [(\mu_v)]T_{[(\mu^h)^{-1}]} \in \tilde{\Lambda}$, where μ_v is the initial vertical edge of the unique rectangle that has diagonal μ, and so on. One can easily show that $\tilde{\gamma} : \pi_1(\hat{N}, x_0) \to \tilde{\Gamma}; [\mu] \mapsto \tilde{\gamma}_{[\mu]}$ and $\tilde{\lambda} : \pi_1(\hat{N}, x_0) \to \tilde{\Lambda}; [\mu] \mapsto \tilde{\lambda}_{[\mu]}$ are homomorphisms. $\pi_1(\hat{N}, x_0)$ acts naturally from the left on the universal cover of $\hat{N}$. We want to transfer this action to $\tilde{M} \times \tilde{\Sigma}$ via the foliated isomorphism Φ. Let $[\mu] \in \pi_1(\hat{N}, x_0)$ and $\Phi([\tau], [\sigma]) = [\nu]$. Then

$$\Phi^{-1}([\mu][\nu]) = ([(\mu_h)]T_{[(\mu^v)^{-1}]}[\tau], [\mu_v]T_{[(\mu^h)^{-1}]}[\sigma]) = (\tilde{\gamma}_{[\mu]}([\tau]), \tilde{\lambda}_{[\mu]}([\sigma])).$$

This shows that the action of $\pi_1(\hat{N}, x_0)$ respects the natural bifoliation on $\tilde{M} \times \tilde{\Sigma}$. Thus:

Proposition 4.12. *$\pi_1(\hat{N}, x_0)$ is a subgroup of Diff$(\tilde{M}) \times I(\tilde{\Sigma})$. The projection of $\pi_1(\hat{N}, x_0)$ on the first component is $\tilde{\Gamma}$, the one on the second is $\tilde{\Lambda}$.* □

The projection homomorphisms are $\tilde{\gamma}$ and $\tilde{\lambda}$. This describes a new view on the transversal holonomy group, even for the Weyl group of a polar action, for instance for the isotropy representation of a symmetric space. In [Tö] we also show for a bifoliation $(\hat{N}, \hat{\mathcal{F}}, \hat{\mathcal{F}}^\perp)$:

Proposition 4.13. *If $\hat{M}$ is a leaf of $\hat{\mathcal{F}}$ and Σ a leaf of $\hat{\mathcal{F}}^\perp$, then*

$$|\pi_1(\hat{N}, x_0)| = |\pi_1(\hat{M}, x_0)| \cdot |\pi_1(\Sigma, x_0)| \cdot |\hat{M} \cap \Sigma|.$$ □

In the case of infinity, the interpretation of this equation is that the left value is infinity if and only if at least one of the factors on the right side is infinity.

Now we want to focus on $\tilde{\Gamma}$. There is a natural injective representation $\rho_\Sigma : \pi_1(\Sigma, x_0) \to \tilde{\Gamma}; [\sigma] \mapsto [\sigma]$. This means that $\tilde{\Gamma}$ contains the deck transformations of $\pi_\Sigma : \tilde{\Sigma} \to \Sigma; [\sigma] \mapsto \sigma(1)$. Since the elements of $\pi_1(\Sigma, x_0)$ do not contribute to the holonomy action on Σ when restricting them to local transformations on Σ, it is natural to ask when we can divide $\pi_1(\Sigma, x_0)$ out of $\tilde{\Gamma}$. The following lemma gives a geometric condition.

Lemma 4.14. *$\tilde{\Gamma}$ normalizes $\pi_1(\Sigma, x_0)$ if Σ has trivial normal holonomy, i.e. if $T_g = \mathrm{id}_{\tilde{M}}$ for any $g \in \pi_1(\Sigma, x_0)$.*

Proof. Let $\gamma = [\sigma]T_\tau \in \tilde{\Gamma}$ and $g \in \pi_1(\Sigma, x_0)$. Then for any $\lambda \in \tilde{\Sigma}$ we have

$$\begin{aligned} \gamma(g\lambda) = & \; [\sigma]T_{[\tau]}(g\lambda) \\ = & \; [\sigma]T_{[\tau]}g \cdot T_{T_g[\tau]}(\lambda) \\ = & \; [\sigma]T_{[\tau]}g[\sigma]^{-1} \cdot [\sigma]T_{[\tau]}(\lambda) \qquad \text{because } T_g = \mathrm{id}_{\tilde{M}} \\ = & \; g'\gamma(\lambda) \end{aligned}$$

for $g' = [\sigma]T_{[\tau]}g[\sigma]^{-1} \in \pi_1(\Sigma, x_0)$. $\square$

In the case of the lemma, we say that Σ is an *exceptional section*. We remark that this is the generic case if each section is embedded. We call the group

$$\Gamma = \tilde{\Gamma}/\pi_1(\Sigma, x_0)$$

the *transversal holonomy group* of Σ. It is a subgroup of $I(\Sigma)$. The transversal holonomy group generalizes the Weyl group of the isotropy representation of a symmetric space (or polar action) and the fundamental domains of Γ generalize the Weyl chambers. Note that Γ is independent of the choice of the fixed regular point x_0 in the given section. But it depends on the choice of the section Σ, unlike $\tilde{\Gamma}$. Similarly we define the subgroup $\Lambda = \tilde{\Lambda}/\pi_1(M, x_0)$ in $\mathrm{Diff}(\tilde{M})$ if M has trivial normal holonomy.

Moreover there is a representation $\rho_M : \pi_1(M, x_0) \to \tilde{\Gamma}; [\tau] \mapsto T_{[\tau]}$ that is in general not injective. Let K_{x_0} be the kernel of ρ_M and $H_{x_0} = \pi_1(M, x_0)/K_{x_0}$. Since the action of $\pi_1(M, x_0)$ on $\tilde{\Sigma}$ by ρ_M is isometric, it is already determined by its infinitesimal (orthogonal) action on $T_{x_0}\tilde{\Sigma} = \nu_{x_0}M$, which is

$$\begin{aligned} \pi_1(M, x_0) \times \nu_{x_0}M & \to \nu_{x_0}M \\ ([\alpha], v) & \mapsto (\overset{1}{\underset{0}{\|}}\alpha)v. \end{aligned}$$

This implies that H_{x_0} is isomorphic to the normal holonomy group of M. Thus we can write $\bar{M} = \tilde{M}/K_{x_0}$ for the normal holonomy principal bundle $\bar{M}$, and H_{x_0} is the group of deck transformations of $\tilde{M} \to \bar{M}$.

Let $\{x_i\}_{i \in I} = M \cap \Sigma$. We define $[\sigma_0] = [c_{x_0}]$ and $[\tau_0] = [c_{x_0}]$. For each $i \in I, i \neq 0$, we choose a horizontal curve $[\sigma_i]$ and a vertical curve $[\tau_i]$ from x_0 to x_i. We can write any element $[\sigma]T_{[\tau]} \in \tilde{\Gamma}$ as

$$\alpha(i, g, h) := g[\sigma_i]T_{h[\tau_i]},$$

where $g \in \pi_1(\Sigma, x_0)$ and $h \in \pi_1(M, x_0)$.

Lemma 4.15. $\alpha(i, g, h) = \alpha(j, g', h') \iff i = j, g = g'$ *and* $h^{-1}h' \in K_{x_0}$.

Proof. ($\Leftarrow$) follows from $T_{hk[\tau_i]} = T_{[\tau_i]} \circ T_k \circ T_h = T_{[\tau_i]} \circ T_h = T_{h[\tau_i]}$ for $k \in K_{x_0}$, since $T_k = \mathrm{id}_{\tilde{\Sigma}}$. For ($\Rightarrow$) apply $[c_{x_0}]$ to both sides. We see that $g[\sigma_i] = \alpha(i,g,h)[c_{x_0}] = \alpha(j,g',h')[c_{x_0}] = g'[\sigma_j]$. The endpoints x_i and x_j are equal, so $i=j$. Thus $g=g'$. It follows $T_h = T_{h'}$, or $T_{h^{-1}h'} = \mathrm{id}_{\tilde{\Sigma}}$. Observe that $h^{-1}h' \in \pi_1(M,x_0)$. Now we have $h^{-1}h' \in K_{x_0}$. □

We remark $\rho_\Sigma(g) = \alpha(0,g,[c_{x_0}])$ for any $g \in \pi_1(\Sigma,x_0)$ and $\rho_M(h) = \alpha(0,[c_{x_0}],h)$ for any $h \in \pi_1(M,x_0)$.

Proposition 4.16. *The set of leaves $N/\mathcal{F}$ is equal to $\tilde{\Sigma}/\tilde{\Gamma}$. If Σ is not an exceptional section, Γ is defined and preserves $\mathcal{F}$ (specified in the proof). Moreover $N/\mathcal{F} = \Sigma/\Gamma$. The set of sections is $\tilde{M}/\tilde{\Lambda}$.*

Proof. We have seen at the beginning of this section, that $\hat{\pi}$ defines a bijection between the set of leaves of $\mathcal{F}$ and that of $\hat{\mathcal{F}}$. Therefore $N/\mathcal{F} = \hat{N}/\hat{\mathcal{F}}$. As justified before we can identify Σ with the leaf $T\Sigma$ of $\hat{\mathcal{F}}^\perp$ in $\hat{N}$. Under this identification, the section $i_{x_0} : \Sigma \to N$ with image Σ' is equal to the restriction $\hat{\pi} : T\Sigma \to \Sigma'$ of the footpoint map $\hat{\pi}$ to $T\Sigma$. We also identify the actions of $\tilde{\Gamma}$ with respect to $\mathcal{F}$ and $\hat{\mathcal{F}}$. For the bifoliated manifold $\hat{N}$ we observe that

$$\tilde{\Gamma}[\sigma] = \pi_\Sigma^{-1}(\hat{M}_{\sigma(1)} \cap T\Sigma)$$

for any $[\sigma] \in \tilde{\Sigma}$ and hence $\hat{N}/\hat{\mathcal{F}} = \widetilde{T\Sigma}/\tilde{\Gamma} = \tilde{\Sigma}/\tilde{\Gamma}$. Assume that Σ is not exceptional, so Γ is defined. From the formula above we have $\Gamma(T_x\Sigma) = \hat{M}_x \cap T\Sigma$ (hence $\hat{N}/\hat{\mathcal{F}} = T\Sigma/\Gamma = \Sigma/\Gamma$) and therefore with the obvious identifications $\Gamma(x) = M_x \cap \Sigma$ for regular x. We say that Γ preserves the restriction $\mathcal{F}_r$ of $\mathcal{F}$ to the regular stratum. The last formula is not well-defined for singular $x \in \Sigma'$, since Σ can have self-intersections in x, thus there is no canonical element in the preimage of x in $\Sigma = T\Sigma$ under i_{x_0}. It is true, however, that any two points $V_1, V_2 \in T\Sigma \subset \hat{N}$ with $\hat{\pi}(V_i) = x$ lie in the same leaf, namely $\hat{M}_x$, as seen before. Thus both Γ-orbits coincide and are equal to $\hat{M}_x \cap T\Sigma$. Its image under $\hat{\pi}$ is $M_x \cap \Sigma'$. In this sense Γ preserves $\mathcal{F}$, even in singular points. □

The description of a $\tilde{\Gamma}$-orbit in the proof implies in particular that each element of $\tilde{\Gamma}$ permutes the set $\{g[\sigma_i] \mid i \in I, g \in \pi_1(\Sigma,x_0)\}$. In other words, this defines a representation of $\tilde{\Gamma}$ as a permutation group. This representation is faithful if M has trivial normal holonomy, because of $K_{x_0} = 1$ and Lemma 4.15.

Now let $\mathcal{F}$ be a *proper* singular Riemannian foliation admitting sections. Then each regular leaf M has parallel focal structure and finite normal holonomy. The set $\{x_i\}$ is disrete and closed. We call

$$\mathcal{D}_{x_i} = \{q \in \Sigma \mid d(x_i,q) < d(x_j,q) \text{ for all } j \neq i\}$$

a *Dirichlet region* of the set $\{x_i\}$, where d is the distance function in Σ. These sets are open and disjoint and we have $\bigcup_i \overline{\mathcal{D}}_{x_i} = \Sigma$. The set $\mathcal{D}_{x_i}$ is star-shaped and therefore 1-connected; thus the universal covering $\pi_\Sigma : \tilde{\Sigma} \to \Sigma$ is trivial over $\mathcal{D}_{x_i}$ and we denote the connected component of $\pi_\Sigma^{-1}(\tilde{\mathcal{D}}_{x_i})$ containing $g[\sigma_i], g \in \pi_1(\Sigma,x_0)$ by $\tilde{\mathcal{D}}_{g[\sigma_i]}$. Then $\{\tilde{\mathcal{D}}_{g[\sigma_i]} \mid g \in \pi_1(\Sigma,x_0), i \in I\}$ is the set of Dirichlet regions for $\pi_\Sigma^{-1}(M \cap \Sigma)$.

Proposition 4.17. *Let $\mathcal{F}$ be a proper singular Riemannian foliation admitting sections. Then the action of $\tilde{\Gamma}$ on $\tilde{\Sigma}$ is properly discontinuous. It acts transitively on the set of Dirichlet regions $\{\tilde{\mathcal{D}}_{g[\sigma_i]} \mid g \in \pi_1(\Sigma,x_0), i \in I\}$ and simply transitive,*

if M has trivial normal holonomy. The same holds for Γ, Σ and $\{\mathcal{D}_{x_i}\}_{i\in I}$ if Σ is not an exceptional section. The set of leaves $\tilde{\Sigma}/\tilde{\Gamma}$ is an orbifold.

Proof. The action of $\tilde{\Gamma}$ on $\tilde{\Sigma}$ is isometric and has discrete orbits, thus it is properly discontinous, i.e., for any compact subset K of $\tilde{\Sigma}$ the intersection $\phi(K)\cap K$ is non-empty for only a finite number of $\phi \in \tilde{\Gamma}$. This implies that the set of leaves is an orbifold. The rest follows from Lemma 4.15. □

Remark 4.18. Singular leaves of $\mathcal{F}$ lift to exceptional leaves. Therefore the nonregular points of the orbifold $\tilde{\Sigma}/\tilde{\Gamma}$ correspond exactly to leaves of $\mathcal{F}$ that are either exceptional or singular.

The following lemma is clear.

Lemma 4.19. *The isotropy group $\tilde{\Gamma}_{[c_{x_0}]} = \rho_M(\pi_1(M, x_0)) \cong H_{x_0}$ is characterized in $\tilde{\Gamma}$ by mapping $\tilde{\mathcal{D}}_{[c_{x_0}]}$ onto itself. Consequently $\tilde{\Gamma}_{[\sigma]} \subset \tilde{\Gamma}_{[c_{x_0}]}$ for any $[\sigma] \in \tilde{\mathcal{D}}_{[c_{x_0}]}$. An analogous property holds for Γ if defined.* □

Remark 4.20. Let G be a Riemannian transformation group of (N, g) and let S be a slice through a point $x \in N$ of an orbit Gx. It is known that $G_y \subset G_x$ for every $y \in S$. If Gx is an orbit of maximal dimension, this means that the orbit type of Gx is smaller or equal to that of nearby orbits. This corresponds in our theory to $\Gamma_y \subset \Gamma_x$ and that M_y is covering of M_x.

$H_{x_0} \cong \Gamma_{x_0}$ means that the normal holonomy of a leaf is just the isotropy group of the larger action Γ. In other words, transversal holonomy generalizes normal holonomy of leaves.

We will now give an application for the action of $\tilde{\Gamma}$. Reinhart showed in [Rei] that the nearby leaves of a leaf M in a Riemannian foliation are coverings of M. The next proposition describes the maximal neighborhood for which this is true. Compare with the proof in [Rei].

Proposition 4.21. *Let M be a regular leaf of a proper singular Riemannian foliation $\mathcal{F}$ admitting sections and let $x_0 \in M$ be arbitrary. Then any regular leaf M' through $\mathcal{D}_{x_0}$ covers M and the degree is equal to the holonomy orbit $\tilde{\Gamma}_{[c_{x_0}]}[\gamma]$, where γ is a shortest geodesic in Σ from x_0 to a point in M'.*

Proof. Let $y_0 \in \mathcal{D}_{x_0} \cap M'$ and let γ_0 be a shortest geodesic from x_0 to y_0 which is contained in $\mathcal{D}_{x_0}$. Moreover let $Y := \{[\gamma_j]\}_{j\in J} := \tilde{\Gamma}_{[c_{x_0}]}[\gamma_0]$. We define an action $h \cdot ([\tau], [\gamma_j]) = (h[\tau], T_{[h^{-1}]}[\gamma_j])$ of $\pi_1(M, x_0)$ on $\tilde{M} \times Y$. Note that this group acts from the left by Lemma 4.10, and the action is free and properly discontinuous. Let $\tilde{M} \times_{\pi_1(M,x_0)} Y := (\tilde{M} \times Y)/\pi_1(M, x_0)$ We want to show that

$$\begin{array}{ccc} \tilde{M} \times_{\pi_1(M,x_0)} Y & \to & M' \\ & & \\ ([\tau], [\gamma_j]) & \mapsto & (T_\tau \gamma_j)(1) \end{array}$$

is a diffeomorphism. The map is clearly surjective. We show that it is well-defined. Let $h \in \pi_1(M, x_0)$ and $([\tau], [\gamma_j]) \in \tilde{M} \times Y$. Then

$$(T_{T_{h^{-1}}\gamma_j} h\tau)(1) = (T_{T_{h^{-1}}\gamma_j} h \cdot T_{T_h T_{h^{-1}}\gamma_j} \tau)(1) = (T_{\gamma_j} \tau)(1),$$

so the map is well-defined. We prove injectivity. Let $(T_{\gamma_j}\tau)(1) = (T_{\gamma_k}\tau')(1)$ for vertical curves τ, τ' starting at x_0. Then $\tau(1) = \tau'(1)$, so there is exactly

one $h \in \pi_1(M, x_0)$ such that $h[\tau] = [\tau']$. We claim $\gamma_k = T_{h^{-1}}\gamma_j$. We have $(T_\tau\gamma_j)(1) = (T_{\gamma_j}\tau)(1) = (T_{\gamma_k}\tau')(1) = (T_{\tau'}\gamma_k)(1)$. Thus $(T_\tau\gamma_j)(1) = (T_{h\tau}\gamma_k)(1) = (T_\tau(T_h\gamma_k))(1)$. Applying $T_{\tau^{-1}}$ shows $\gamma_j(1) = (T_h\gamma_k)(1)$, i.e., $\pi_\Sigma([\gamma_j]) = \pi_\Sigma(T_h[\gamma_k])$. Since $[\gamma_j]$ and $T_h[\gamma_k]$ lie in $\widetilde{\mathcal{D}}_{[c_{x_0}]}$ this implies $[\gamma_j] = T_h[\gamma_k]$ and we proved our claim. Now the above map is a diffeomorphism. Thus $M' = \tilde{M} \times_{\pi_1(M,x_0)} Y$ covers M with typical fiber Y. □

With the introduced methods we have extended Proposition 2.4 in [Tö]:

Proposition 4.22. *Let M be a closed and embedded submanifold with parallel focal structure and finite normal holonomy. If $v \in \nu M$ is a multiplicity k focal normal of vertical type so are its normal parallel translations. In other words the vertical focal data is also invariant under normal parallel translation. If v is a cut normal, so are its normal parallel translations. In particular, the cut distance function is constant along the parallel normal fields.* □

The last statement is already known from Proposition 2.4. With this proposition we can distinguish a submanifold with parallel focal structure from other submanifolds by its cut locus. The next proposition proved in [Tö] shows the relation between $\{\mathcal{D}_{x_i}\}$ and the cut locus of a regular leaf, which has parallel focal structure as we know. It also shows that a leaf M' through $\mathcal{D}_{x_0}$ is regular if M has a globally flat normal bundle.

Proposition 4.23. $\bigcup_{i\in I} \partial\mathcal{D}_{x_i} \subset \mathcal{C}_{(M,N)}$ *and* $\mathcal{C}_{(M,N)} \cap \mathcal{D}_{x_i}$ *are points of the cut locus of* $\exp^\Sigma_{x_i}$*. If M has a globally flat normal bundle, then the focal points of horizontal type are contained in* $\bigcup_{i\in I} \partial\mathcal{D}_{x_i}$*.* □

Now we express Proposition 4.21 as a corollary in terms of the cut locus.

Corollary 4.24. *Let M be a closed and embedded submanifold with parallel focal structure and finite normal holonomy. Then the parallel submanifolds that are not contained in the cut locus of M are coverings of M.* □

The following result is a another corollary of Proposition 4.21.

Corollary 4.25. *The regular leaves with a globally flat normal bundle are diffeomorphic to each other. They cover any other regular leaf, the exceptional leaves. These exceptional leaves, if they exist, are contained in the cut locus of any regular leaf with a globally flat normal bundle. The union of regular leaves with a globally flat normal bundle is open and dense in N.* □

The exceptional leaves lie in the cut locus of the leaves with trivial normal holonomy. Can we give a sufficient condition on N that guarantees that all regular leaves have trivial normal holonomy? We will show that there are no exceptional leaves, if the ambient space is a simply connected symmetric space. We need some preparations. For a point $p \in N$ we define $\mathcal{P} = \mathcal{P}(N, \varphi \times p)$ as the set of pairs (x, γ), where $x \in M$ and $\gamma : [0,1] \to N$ is a H^1-curve in N with $\gamma(0) = \varphi(x)$ and $\gamma(1) = p$. We write $\mathcal{P}(N, M \times p)$ for the path space if $\varphi : M \to N$ is the inclusion map. It is known that $\mathcal{P}$ is a Hilbert manifold. The smooth function

$$E_p : \mathcal{P} \to \mathbb{R};\ (x,\gamma) \mapsto \int_0^1 \|\dot\gamma(t)\|^2\, dt$$

is called the *energy functional* (associated to p). The map E_p is a Morse function, i.e., it has only non-degenerate critical points, if and only if p is not a focal point of φ. We assume that p is not a focal point, i.e., p is regular for the normal exponential map of M. The energy functional is bounded below by zero and it is known that it satisfies the Palais-Smale condition. For $s \in \mathbb{R}$ we write $\mathcal{P}^s = E_p^{-1}\{[0,s]\}$ and $\mathcal{P}^{s-} = E_p^{-1}\{[0,s)\}$. Let s be a regular value of E_p. The Morse inequalities state $b_k(\mathcal{P}^s) \le \mu_k(E_p|\mathcal{P}^s)$, where $b_k(\mathcal{P}^s)$ is the k-th Betti number of $\mathcal{P}^s$ with respect to $\mathbb{Z}_2$ and $\mu_k(E_p|\mathcal{P}^s)$ is the number of critical points of index k of E_p below s.

Theorem 4.26. *Let $\varphi : M \to N$ be a proper immersion with parallel focal structure with finite normal holonomy into a simply connected symmetric space N. Then φ factorizes finitely over an embedding that has a globally flat normal bundle. There are no exceptional parallel submanifolds and the cut locus of M only consists of focal points.*

The first part of the next result was proven in Lemma 1A.3 of [PoTh] for polar actions. The proof refers to Lemma 2.10 in [Ew], which has a gap. We want to present the complete proof and relate the occurrence of exceptional leaves to the cut locus. It is well-known that a closed hypersurface M of a simply connected manifold N is orientable and thus has a globally flat normal bundle. If the codimension is greater than one we have to argue differently.

Proof. Since the image of φ is a leaf and because φ is proper, it factorizes finitely over an embedding. So we can assume that φ is this embedding. Let p be a regular point in N with respect to the normal exponential map of M. We claim that E_p has only one (local) minimum. We will prove this later. Assume now that there is an exceptional parallel manifold M' of M. We choose $\varepsilon > 0$ smaller than the injectivity radius of M', which is positive since the cut distance function is constant, $p \in M'$ and $v \in \nu_p M'$ with non-trivial holonomy degree and $\|v\| < \varepsilon$. Let $w \neq v$ in $\nu_p M'$ be a normal parallel translation of v. Let $M := M'_v$. The geodesics $\gamma_v|[0,1]$ and $\gamma_w|[0,1]$, if parameterized in reverse direction, are normal to M, nonfocal and of index 0 by the choice of ε. Therefore there are at least two minima of $E_p : \mathcal{P}(N, M \times p) \to \mathbb{R}$, contradiction. Now assume that there is point p in the cut locus of M that is not a focal point. By Proposition 2.3 it is the endpoint of two minimal normal geodesics of index 0, contradiction.

We will now prove the claim. First we observe that $\mathcal{P}$ is connected because of the homotopy sequence of the fibration $\mathcal{P} \to M; c \mapsto c(0)$ and $\pi_1(N) = 1$. Let $\gamma \in \mathcal{P}$ be an arbitrary critical point of index 1 of E_p, i.e., γ is a normal geodesic of index 1, with $E_p(\gamma) = \kappa$. Let $\varepsilon > 0$ so small that there is no critical level in $[\kappa - \varepsilon, \kappa + \varepsilon]$ other than κ. Let e^1 be the corresponding 1-cell in $\mathcal{P}^{\kappa+\varepsilon}$ through γ attached to $\mathcal{P}^{\kappa-\varepsilon}$. Let $v \in \nu_x M, x = \gamma(0)$ with $\gamma_v = \gamma$, and let $t_0 v$, $0 < t_0 < 1$ be the focal normal with multiplicity 1. First we assume that v is a focal normal of horizontal type. Let F be the focal leaf of v in M (see Remark 4.4) and not in $\bar{M}$. Since F is 1-dimensional and compact we have $F \cong S^1$. We construct a variation $\lambda : F \to \mathcal{P}$ of γ by

$$\lambda(y)(t) := \begin{cases} \eta(tv_y) & \text{if } t \in [0, t_0] \\ \gamma(t) & \text{if } t \in [t_0, 1] \end{cases}$$

(compare with [Th1] and [Ew]). This smooth map is injective (if we took the focal leaf in $\bar{M}$ this map would only be a covering) and Ewert deforms it under the negative gradient flow of E_p to a map $\lambda' : F \to \mathcal{P}$ that has a unique non-degenerate

maximum in x. We denote the generator of $H_1(F) = H_1(S^1)$ by $[S^1]$, where we consider homology over $\mathbb{Z}_2$. Then $z := \lambda_*([S^1]) = \lambda'_*([S^1]) \in H_1(\mathcal{P}^{\kappa+\varepsilon})$ is a so-called *Bott-Samelson cycle* with the property $j_1(z) = [e^1]$, where $j_1 : H_1(\mathcal{P}^{\kappa+\varepsilon}) \to H_1(\mathcal{P}^{\kappa+\varepsilon}, \mathcal{P}^{\kappa-\varepsilon})$ is the map of the homology sequence of the pair $(\mathcal{P}^{\kappa+\varepsilon}, \mathcal{P}^{\kappa-\varepsilon})$. Now we assume that v is a focal normal of vertical type, i.e., $\gamma(t_0)$ is conjugate to x along γ in Σ_x with multiplicity 1. Since Σ_x is a symmetric space as a totally geodesic submanifold of N, an S^1-action fixing x and $\gamma(t_0)$ applied to $\gamma|[0, t_0]$ gives an S^1-familiy of geodesics from $\gamma(0)$ to $\gamma(t_0)$. We extend this variation as above to a map $\lambda : S^1 \to \mathcal{P}$ and Ewert proves that also $z := \lambda_*([S^1]) \in H_1(\mathcal{P}^{\kappa+\varepsilon})$ is a Bott-Samelson cycle with $j_1(z) = [e^1]$. Now let us consider a part of the homology sequence of the pair $(\mathcal{P}^{\kappa+\varepsilon}, \mathcal{P}^{\kappa-\varepsilon})$:

$$H_1(\mathcal{P}^{\kappa+\varepsilon}) \xrightarrow{j_1} H_1(\mathcal{P}^{\kappa+\varepsilon}, \mathcal{P}^{\kappa-\varepsilon}) \xrightarrow{\partial_1} H_0(\mathcal{P}^{\kappa-\varepsilon}) \xrightarrow{i_0} H_0(\mathcal{P}^{\kappa+\varepsilon})$$

By standard Morse theory the set of $[e^1]$ for the descending cells e^1 of all critical points γ of index 1 on level κ is a basis of $H_1(\mathcal{P}^{\kappa+\varepsilon}, \mathcal{P}^{\kappa-\varepsilon})$. As seen above, each $[e^1]$ is in the image of j_1. Therefore $\partial_1 = 0$. Thus $i_0 : H_0(\mathcal{P}^{\kappa-\varepsilon}) \to H_0(\mathcal{P}^{\kappa+\varepsilon})$ is injective for every critical level κ and consequently also $i_0 : H_0(\mathcal{P}^r) \to H_0(\mathcal{P})$ for every r. Since $\mathcal{P}$ is connected, $H_0(\mathcal{P}) = \mathbb{Z}_2 = H_0(\mathcal{P}^r)$, so $\mathcal{P}^r$ is connected for every r. If E_p had at least two minima, the higher one say on level κ, $\mathcal{P}^{\kappa+\varepsilon}$ would be disconnected for small ε, contradiction. $\square$

Note that we can drop symmetry of N if we assume the sections are either symmetric or do not have conjugate points.

References

[A] M. Alexandrino: *Singular Riemannian Foliations with Sections*, Submitted article, see also http://arxiv.org math.DG/0311454.

[Bou] H. Boualem: *Feuilletages riemanniens singuliers transversalement integrables*, Composito Mathematica **95** (1995), 101-125.

[BH1] R.A. Blumenthal, J. Hebda: *De Rham Decomposition Theorem for Foliated Manifolds*, Ann. Inst. Fourier **33**(2) (1983) 133-198.

[BH2] R.A. Blumenthal, J. Hebda: *Ehresmann Connections for Foliations*, Indiana Mathematical Journal **33** (1984) 597-611.

[Ew] H. Ewert: *Equifocal Submanifolds in Riemannian Symmetric Spaces*, Doctoral Dissertation, Universität zu Köln, 1998.

[HLO] E. Heintze, X. Liu, C. Olmos: *Isoparametric Submanifolds and a Chevalley-type Restriction Theorem*, Preprint.

[Kl] W. Klingenberg: *Riemannian Geometry*, de Gruyter, Berlin 1982.

[Mo] P. Molino: *Riemannian Foliations*, Progress in Mathematics Vol. 73, Birkhäuser, Boston 1988.

[Pa] R.S. Palais: *A Global Formulation of the Lie Theory of Transformation Groups*, Mem. Amer. Math. Soc. **22** (1957)

[PaTe] R.S. Palais, C.L.Terng: *Critical Point Theory and Submanifold Geometry*, Geometry Lecture Notes in Math., Vol. 1353, Springer, Berlin 1988.

[PoTh] F. Podestà, G. Thorbergsson: *Polar Actions on Rank One Symmetric Spaces*, J. Diff. Geom. **53** (1999) no. 1, 131-175

[Rei] B. Reinhart: *Foliated Manifolds with Bundle-like Metrics*, Ann. of Math. (2) **69** (1959), 119-132

[Th1] G. Thorbergsson: *Dupin Hypersurfaces*, Bull. London Math. Soc. **15** (1983), 493-498.

[Th2] G. Thorbergsson: *A Survey on Isoparametric Hypersurfaces and their Generalizations*, Handbook of Differential Geometry, Vol. 1, Elsevier Science B.V. 2000.

[TeTh] C.L. Terng, G. Thorbergsson: *Submanifold Geometry in Symmetric Spaces*, J. Diff. Geom. **42** (1995), 665-718.

[Tö] D. Töben: *Submanifolds with Parallel Focal Structure*, Doctoral Disseration, Universität zu Köln, 2003.

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